DESIGN OF IIR DIGITAL FILTERS & REALIZATIONS

Introduction:

Basically a digital filter is a Linear time-invarient discrete-time system. The terms Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) are used to distinguish filter types.

FIR filters : FIR filters are of non-recursive type, where by the present output depends on only past and present inputs

IIR filters: : IIR filters are recursive type, where by the present output depends on past outputs, past and present inputs.

The impulse response h(n) for a realizable filter is

$$h(n) = 0 \quad \text{for } n \leq 0$$

and for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

IIR filters have the transfer function of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \text{--- (1)}$$

The design of an IIR filter for the given specifications is to find filter co-efficients a_ks and b_ks in eq (1)

Frequency selective Filters

A filter is one, which rejects unwanted frequencies from the input signal and allow the desired frequencies.

The range of frequencies of signal that are passed through the filter is called passband and those frequencies that are blocked is called stopband.

The filters are of different types.

1. Lowpass filter
2. Highpass filter
3. Bandpass filter
4. Band Reject filter.

1. Lowpass filter

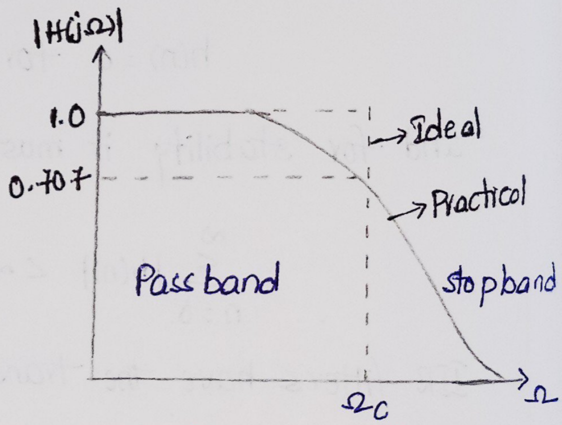
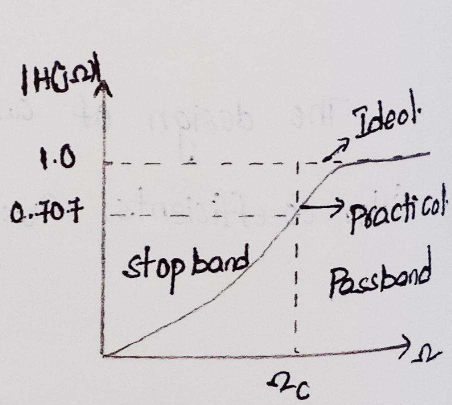


Fig: Magnitude response of LPF

The magnitude response of an ideal low pass filter allows low frequencies in the pass band $0 < \Omega < \Omega_c$ to pass, whereas the higher frequencies in the stopband $\Omega > \Omega_c$ are blocked. The frequency Ω_c between two bands is the cut-off frequency, where the magnitude $|H(j\Omega)| = 1/\sqrt{2}$

2. High Pass filter



Fig(b): Magnitude response of HPF

The highpass filter allows high frequencies above $\Omega > \Omega_c$ and rejects the frequencies between $\Omega = 0$ and $\Omega = \Omega_c$. The magnitude response of an ideal and practical highpass filter is shown in fig(b).

3. Band Pass filter

It allows only a band of frequencies Ω_1 to Ω_c to pass and stops all other frequencies. The ideal and practical response of bandpass filter are shown in fig 1(c) & 1(d)

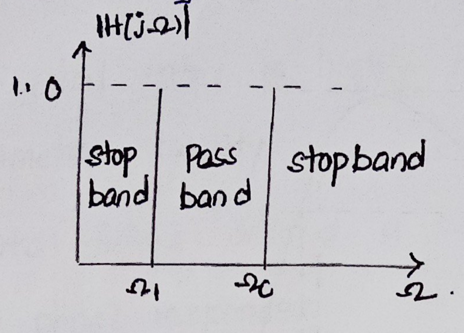


Fig 1(c)
Magnitude response of Ideal Band pass filter

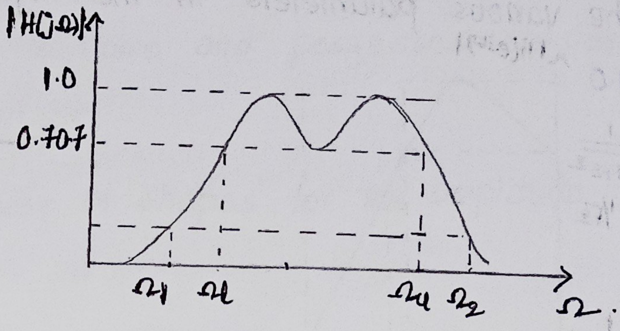


Fig 1(d).
Magnitude response of Practical BPF

4. Band reject filter

It rejects all the frequencies between Ω_1 and Ω_2 and allows remaining frequencies. The magnitude response of an ideal and Practical filters is shown in fig 1(e) & 1(f).

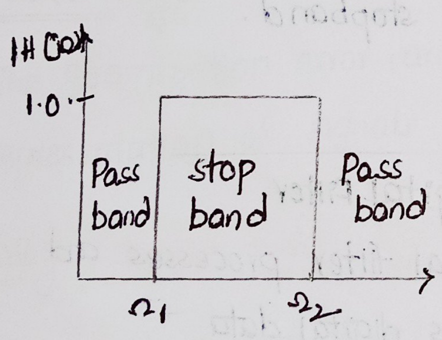


Fig 1(e): Magnitude response of Ideal Band reject filter

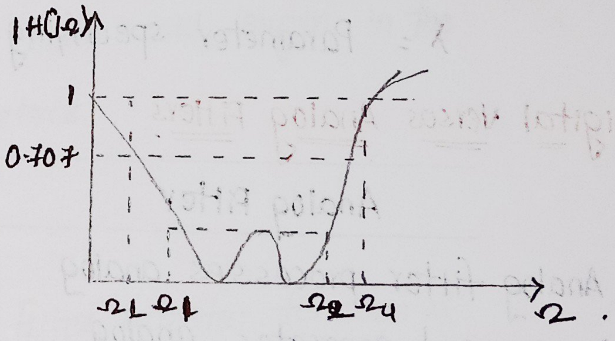


Fig 1(f): Magnitude response of Practical Band reject filter.

→ Design of Digital filters from Analog filters

1. Map the desired digital filter specifications into those for an equivalent analog filter
2. Derive the analog transfer function for the analog prototype

3. Transform the transfer function of the analog prototype into an equivalent digital filter transfer function.

Fig 2(a) shows the magnitude response of a digital lowpass filter.

The various parameters in the figure are

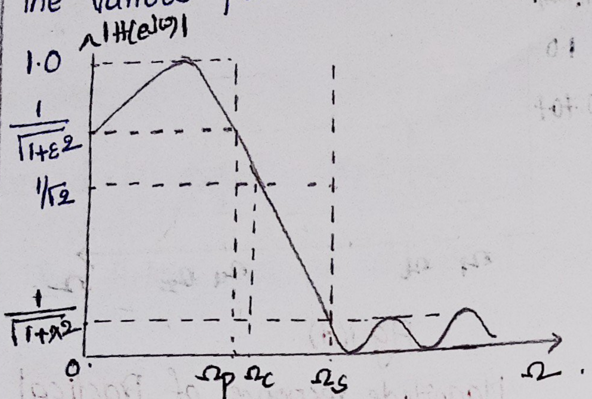


Fig 2: specifications for the magnitude response of Analog LPF

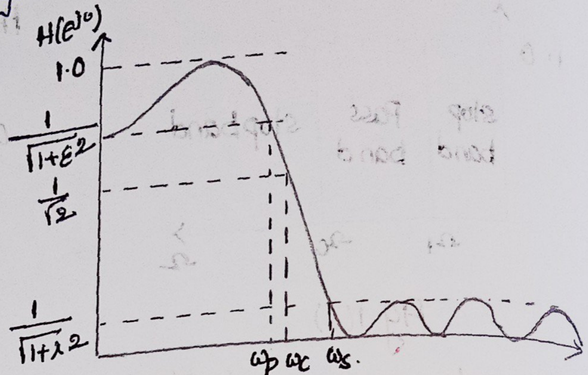


Fig 2(a): specifications for the magnitude response of Digital LPF

where ω_p = Passband frequency in radians

ω_s = stopband frequency in radians

ω_c = 3-dB cut-off frequency in radians

ε = Parameter specifying allowable passband

λ = Parameter specifying allowable stopband.

→ Digital Versus Analog Filters

Analog Filter	Digital Filter
1. Analog filter processes analog inputs and generates analog outputs.	1. A Digital filter processes and generates digital data.
2. Analog filters are constructed from active or passive electronic components.	2. A digital filter consists of elements like Adder, multiplier and delay unit
3. Analog filter is described by a difference equation.	3. Digital filter is described by a difference equation.
4. The frequency response of an analog filter can be modified by changing the components.	4. The frequency response can be changed by changing the filter co-efficients.

Advantages and Disadvantages of Digital filters

Advantages

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature and power supply variations.
2. A digital filter is highly immune to noise and possesses considerable parameter stability.
3. Digital filters afford a wide variety of shapes for the amplitude and phase responses.
4. There are no problems of input or output impedance matching with digital filters.
5. Digital filters can be operated over a wide range of frequencies.
6. The co-efficients of digital filter can be programmed and altered any time to obtain the desired characteristics.
7. Multiply filtering is possible only in digital filter

Disadvantage

1. The quantization error arises due to finite word length in the representation of signals and parameters.

Analog Lowpass Filter Design

The most general form of analog filter transfer function is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

where $H(s)$ is the Laplace transform of the impulse response of $h(t)$,

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

For a stable analog filter, the poles of $H(s)$ lie in the left half of the s-plane. There are two types of analog filter design. They are :

1. Butterworth filter
2. Chebyshev filter.

→ **Analog lowpass Butterworth Filter**

The magnitude function of the Butterworth lowpass filter is given by

$$H(j\Omega) = \frac{1}{\left[1+(\Omega/\Omega_c)^{2N}\right]^{1/2}} \quad N = 1,2,3 \ldots \quad -①$$

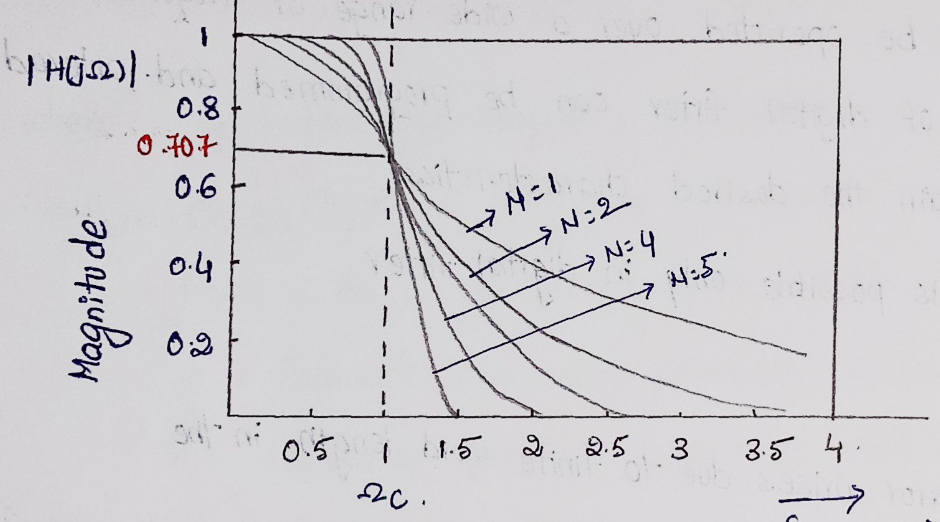


Fig 3 : Lowpass Butterworth magnitude response

Where N is the order of the filter and Ω_c is the cut-off frequency.

As shown in fig3 the function is monotonically decreasing, where the maximum response is unity at $\Omega=0$. The ideal response is shown by the dash line.

It can be seen that the magnitude response approaches the ideal low pass characteristics as the order 'N' increases.

For values $\Omega < \Omega_c$, $|H(j\Omega)| = 1$ for values $\Omega > \Omega_c$, the values of $|H(j\Omega)|$ decreases rapidly. At $\Omega = \Omega_c$, the curves passes through 0.707 which corresponds to -3dB point.

From eq ①, we can get magnitude square function of a normalized Butterworth filter (to 1 rad/sec cutoff frequency) as

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}} \quad \text{where } N = 1,2,3 \dots \quad -②$$

Now, let us derive the transfer function of a stable filter. For this purpose, substituting $\Omega = \frac{s}{j}$, we can write eq ② as

$$|H(j\Omega)|^2 = H(-\Omega^2) = H\left(\left(\frac{s}{j}\right)^2\right)$$

$$= H(-s^2)$$

$$|H(j\Omega)|^2 = H(s).H(-s)$$

eq ② can be written as

$$H(s).H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} = \frac{1}{1+(-1)^N s^{2N}} = \frac{1}{1+(-s^2)^N} \quad -③$$

The above relation tell us that, this function has poles in the Left half of s-plane (LHP) as well as Right half of the s-plane, because of the presence of two factors $H(s)$ and $H(-s)$

$$1+(-s^2)^N = 0 \quad -④$$

For N is odd, the above eq becomes

$$-s^{2N} = -1$$

$$s^{2N} = 1 = e^{j2\pi k}$$

$$s_k = e^{j\pi k/N} \quad k = 1,2 \dots N \quad -⑤$$

For N even, the eq ④ becomes

$$s^{2N} = -1 = e^{j(2k-1)\pi}$$

$$S_k = e^{j(2k-1)\pi/2N} \quad \text{for } k = 1, 2 \ldots 2N. \quad -⑥$$

Let $N=3$ then eq ④ becomes $s^6 = 1$

We know that for N is odd, then roots can be obtained from eq⑤ ie

$$S_1 = e^{j\pi.1/3} = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j0.866$$

$$S_2 = e^{j\pi.2/3} = e^{j2\pi/3} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j0.866$$

$$S_3 = e^{j\pi 3/3} = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$S_4 = e^{j\pi.4/3} = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j0.866$$

$$S_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j0.866$$

$$S_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1.$$

All the above poles are located in the s-plane as shown in fig 4

It is found that the angular separation between the poles is given by $\frac{360}{2N} = 60°$ and all poles lie on the unit circle.

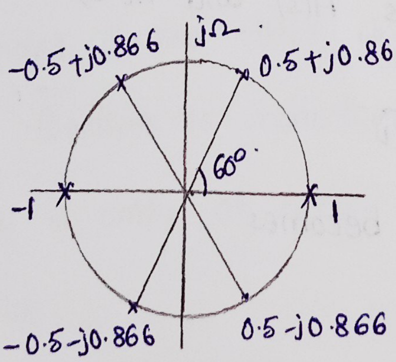


Fig: Pole locations in the s-plane for magnitude square function of Butterworth filter.

To ensure stability and considering only the poles that lie in the left half of the s-plane, we can write the denominator of the transfer function H(s) as

$$(s+1)\left((s+0.5)^2+(0.866)^2\right) = (s+1)(s^2+s+1)$$

Therefore, the transfer function of a 3rd order Butterworth filter for cut-off frequency $\Omega_c = 1 \text{ rad/sec}$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

As we are interested on the poles, which lies in the left half of the s-plane, the same can be found by using the formula $s_k = e^{j\phi_k}$ where

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2 \ldots N \qquad - (7)$$

Let N=4 (i.e even) the poles can be found from eq (6) as

$$s_1 = e^{j\phi_1} = e^{j\left(\frac{\pi}{2}+\frac{\pi}{8}\right)} = e^{j\frac{4\pi+\pi}{8}} = e^{j\left(\frac{5\pi}{8}\right)} = \cos\frac{5\pi}{8} + j\sin\frac{5\pi}{8} = -0.3827 + j0.9239$$

$$s_2 = e^{j7\pi/8} = -0.9239 + j0.3827.$$

$$s_3 = e^{j9\pi/8} = -0.9239 - j0.3827$$

$$s_4 = e^{j11\pi/8} = -0.3827 - j0.9239$$

Now the denominator of transfer function H(s) is

$$\left((s+0.3827)^2+(0.9239)^2\right)\left((s+0.9239)^2+(0.3827)^2\right)$$

$$= (s^2+1.847765+1)(s^2+0.765365+1)$$

For fourth order Butterworth filter the transfer function for $\Omega_c = 1 \text{ rad/sec}$ is given by

$$H(s) = \frac{1}{(s^2+1.847765+1)(s^2+0.76535+1)}$$

The following table gives Butterworth polynomials for various values of N for $\Omega_c = 1$ rad/sec.

N	Denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s^2+1)(s^2+0.61803s+1)(s^2+0.51765s+1)$

The eq ⑦

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \ldots N$$

gives us the pole locations of Butterworth filter for $\Omega = 1$ rad/sec and are known as normalized poles.

In general, the un-normalized poles are given by

$$s_k' = \Omega_c s_k$$

The transfer function of such type of Butterworth filter can be obtained by substituting $s \to s/\Omega_c$ in the transfer function of Butterworth filter.

To Determine the order for the given specifications

Let the maximum passband attenuation in positive dB is α_p (<3dB) at passband frequency Ω_p and α_s is the minimum stopband attenuatuation at the stopband frequency Ω_s. Now the magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1+\varepsilon^2(\Omega/\Omega_p)^{2N}\right]^{1/2}} \quad — \quad (8)$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}} \quad — \quad (9)$$

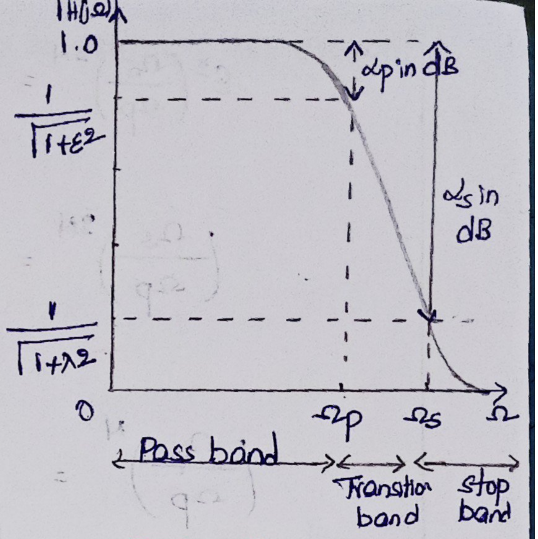


Fig5 : Butterworth approx of Magnitude response

Taking logarithms on both sides. We have

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2(\Omega/\Omega_p)^{2N}\right]$$

From fig (5) We can find that $\Omega = \Omega_p$ the attenuation is equal to α_p

$$20\log|H(j\Omega)| = -\alpha_p = -10\log\left[1+\varepsilon^2\right]$$

$$\alpha_p = 10\log(1+\varepsilon^2)$$

$$0.1\alpha_p = \log(1+\varepsilon^2)$$

$$1+\varepsilon^2 = 10^{0.1\alpha_p} \Rightarrow \varepsilon^2 = 10^{0.1\alpha_p}-1$$

$$\varepsilon = (10^{0.1\alpha_p}-1)^{1/2} \quad — \quad (10)$$

From fig (5) We can ~~fi~~see the minimum stop band attenuation is equal to α_s

At $\Omega = \Omega_s$, the eq (9) becomes.

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}}$$

Taking logarithms on both sides we have

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

$$20\log|H(j\Omega)| = -\alpha_s = -10\log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

$$10^{0.1\alpha_s} = 1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}$$

$$\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s}-1$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{\varepsilon^2} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2}$$

Taking logarithm on both sides

$$N\log\left(\frac{\Omega_s}{\Omega_p}\right) = \log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$\boxed{N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}} \quad \text{— (11)}$$

Since this expression normally does not result in an integer value, we therefore round off 'N' to the next higher integer.

The eq (11) can be written as

$$\boxed{N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}} \qquad \text{where } \lambda = (10^{0.1\alpha_s}-1)^{1/2}$$

$$\varepsilon = (10^{0.1\alpha_p}-1)^{1/2}$$

For simplicity the equation can be written as

$$\boxed{N \geq \frac{\log(A)}{\log(1/k)}} \qquad \text{where } A = \frac{\lambda}{\varepsilon}$$

$$k = \frac{\Omega_p}{\Omega_s}$$

→ Given the specifications $\alpha_p = 1dB$; $\alpha_s = 30dB$; $\Omega_p = 200\ rad/sec$, $\Omega_s = 600\ rad/sec$.

Determine the order of the filter.

Sol. Given : Passband attenuation $= \alpha_p = 1dB$

passband frequency $= \Omega_p = 200\ rad/sec$

stop band attenuation $= \alpha_s = 30dB$

stopband frequency $= \Omega_s = 600\ rad/sec$.

We know that $N \geq \dfrac{\log\left(A\right)}{\log(1/k)}$ where $A = \dfrac{A}{\varepsilon} = \left(\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{1/2}$

$$= \left(\frac{10^{0.1 \times 30} - 1}{10^{0.1 \times 1} - 1}\right)^{1/2} = \left(\frac{10^3 - 1}{10^{0.1} - 1}\right)^{1/2}$$

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

$$= \left(\frac{999}{0.258}\right)^{1/2} = 62.11$$

$$\therefore N \geq \frac{\log(62.11)}{\log(3)} = \frac{1.793}{0.477}$$

$$= 3.758$$

$$= 4$$

→ Prove that $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \dfrac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$.

Proof: The magnitude square function of Butterworth analog LPF is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad - (1)$$

We know that from eq (9) in above topic

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad - (2)$$

Equating eq ① & ② We get

$$1+\left(\frac{\Omega}{\Omega c}\right)^{2N} = 1+\varepsilon^2\left(\frac{\Omega}{\Omega p}\right)^{2N}$$

$$\varepsilon^2\left(\frac{\Omega}{\Omega p}\right)^{2N} = \left(\frac{\Omega}{\Omega c}\right)^{2N}$$

$$\left(\frac{\Omega p}{\Omega c}\right)^{2N} = \varepsilon^2 = 10^{0.1\alpha p}-1$$

$$\frac{\Omega p}{\Omega c} = \left(10^{0.1\alpha p}-1\right)^{1/2N}$$

$$\therefore \Omega c = \frac{\Omega p}{\left(10^{0.1\alpha p}-1\right)^{1/2N}} = \frac{\Omega p}{\varepsilon^{1/N}} \quad - ③$$

But We know

$$\left(\frac{\Omega s}{\Omega p}\right)^{2N} = \frac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}$$

$$\frac{\Omega s}{\Omega p} = \left(\frac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}\right)^{1/2N}$$

$$\Omega s = \Omega p\left(10^{0.1\alpha s}-1\right)^{1/2N} \cdot \frac{1}{\left(10^{0.1\alpha p}-1\right)^{1/2N}} \quad - ④$$

substitute the eq ③ in ④.

$$\Omega s = \Omega c\left[\varepsilon^{1/N}\right] \cdot \frac{\left(10^{0.1\alpha s}-1\right)^{1/2N}}{\left(10^{0.1\alpha p}-1\right)^{1/2N}} \qquad \left[\varepsilon = \left(10^{0.1\alpha p}-1\right)^{1/2}\right]$$

$$= \Omega c\left[\left(\cancel{10^{0.1\alpha p}-1}\right)^{1/2N}\right] \cdot \frac{\left(10^{0.1\alpha s}-1\right)^{1/2N}}{\left(\cancel{10^{0.1\alpha p}-1}\right)^{1/2N}}$$

$$\Omega c = \frac{\Omega s}{\left(10^{0.1\alpha s}-1\right)^{1/2N}} \quad - ⑤$$

Therefore from eq ③ & ⑤

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

→ **steps to design an analog Butterworth LowPass Filter**

1. From the given specifications find the order of the filter 'N'
2. Round off it to the next higher integer
3. Find the transfer function H(s) for $\Omega_c = 1$ rad/sec for the value of N.
4. Calculate the value of cut-off frequency Ω_c
5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting

$s \to \frac{s}{\Omega_c}$ in H(s).

Prev Question → Design an analog Butterworth filter that has a 2dB passband attenuation at a frequency of 20 rad/sec and atleast 10dB stopband attenuation at 30 rad/sec.

Sol Given : Passband attenuation = α_p = 2dB

passband frequency = Ω_p = 20 rad/sec

stopband attenuation = α_s = 10dB

stopband frequency = Ω_s = 30 rad/sec

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log\sqrt{\frac{10^{0.1\times10}-1}{10^{0.1\times2}-1}}}{\log\left(\frac{30}{20}\right)} = \frac{\log\sqrt{\frac{10^1-1}{10^{0.2}-1}}}{\log(1.515)}$$

$$= \frac{0.593}{0.18} = 3.37.$$

Rounding off 'N' to the next higher integer we get

N = 4

The normalized lowpass Butterworth filter for $N=4$ is

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

We know the relation

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.2}-1)^{1/8}} = 21.3868$$

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

$$H(s) = \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.76537\frac{s}{21.3868}+1\right]\left[\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right)+1\right]}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

→ **Analog LowPass Chebyshev Filters**

There are two types of chebyshev filters.

Type I chebyshev filters are all-pole filters that exhibits equiripple behaviour in the passband and a montonic characteristics in the stopband.

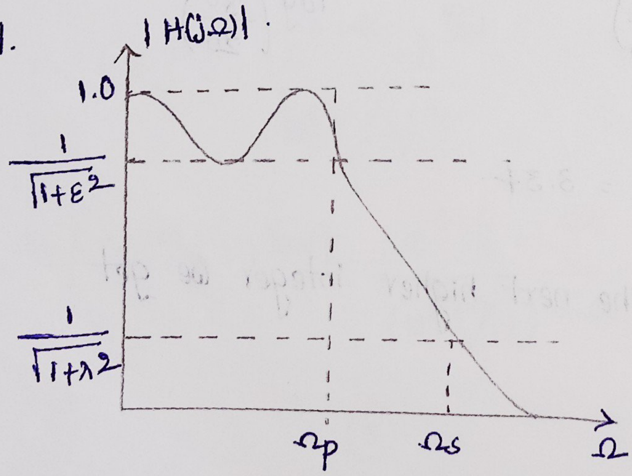


Fig : Type-I chebyshev filter

Type II chebyshev filter contain both poles and zeros and exhibit a monotonic behaviour in the passband and an equi-ripple behaviour in the stop band.

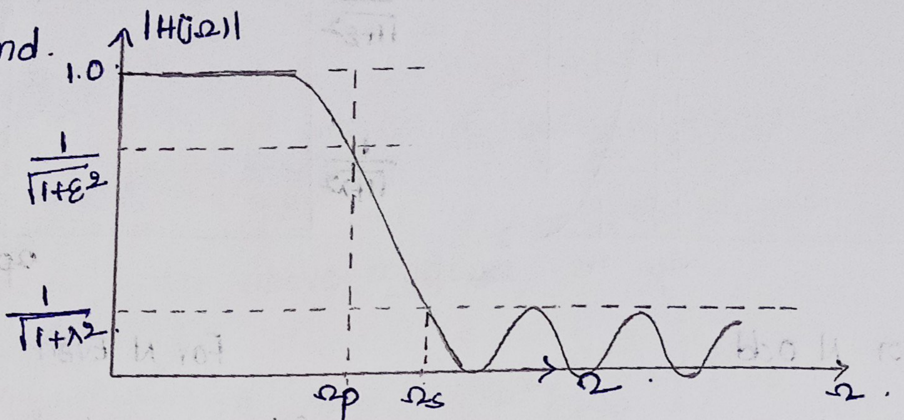


Fig: Type-II chebyshev filter

The magnitude square response of N^{th} order type-I filter can be expressed as

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \quad N = 1, 2 \ldots \quad -①$$

where ϵ is a parameter of the filter related to the ripple in the passband

$C_N(x)$ is the N^{th} order chebyshev polynomial defined as

$$C_N(x) = \cos(N\cos^{-1}x), \quad |x| \leq 1 \quad \text{(pass band)} \quad -②$$

and

$$C_N(x) = \cosh(N\cosh^{-1}x), \quad |x| > 1 \quad \text{(stop band)} \quad -③$$

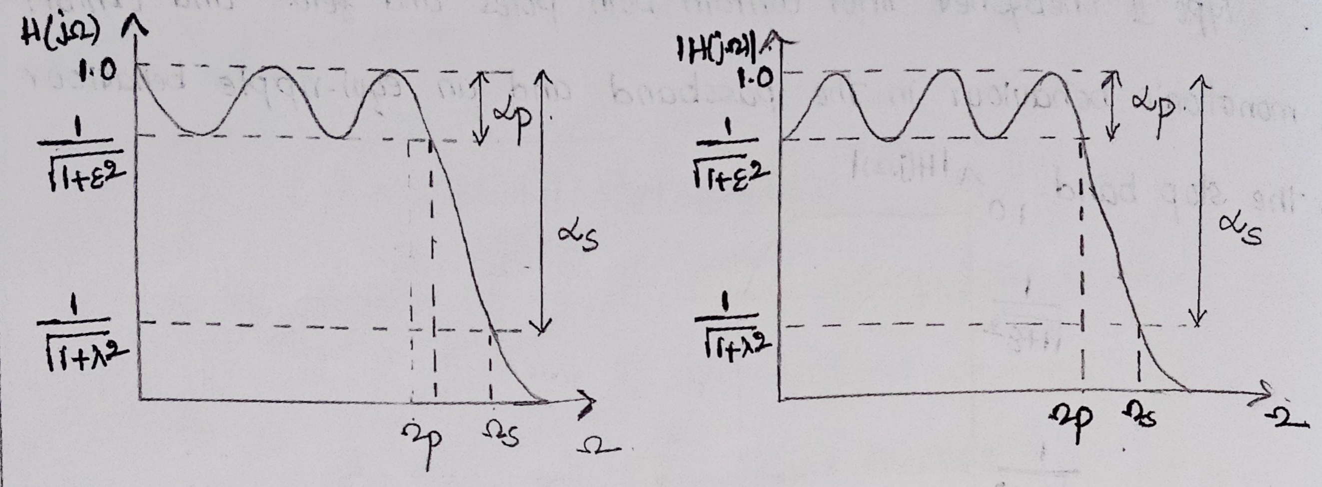
The chebyshev polynomial is defined by the recursive formula

$$C_N(x) = 2x C_{N-1}(x) - C_{N-2}(x), \quad N > 1 \quad -④$$

where $C_0(x) = 1$ and $C_1(x) = x$

Taking logarithm for eq ①, we get

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right] \quad -⑤$$



For N odd | For N Even

Fig: LowPass chebyshev filter magnitude response

Let α_p is the attenuation in positive dB at the passband frequency Ω_p

α_s is the attenuation in positive dB at the stopband frequency Ω_s

At $\Omega = \Omega_p$ the eq ⑤ becomes

$$\alpha_p = 10 \log (1+\varepsilon^2) \qquad (\because C_N(1) = 1) \qquad -⑥$$

which gives $\varepsilon = (10^{0.1\alpha_p} - 1)^{1/2}$. $\quad$ -⑦

At $\Omega = \Omega_s$ the eq ⑤ becomes

$$\alpha_s = 10 \log \left[1 + \varepsilon^2 C_N^2 \left(\frac{\Omega_s}{\Omega_p} \right) \right]$$

$$= 10 \log \left[1 + \varepsilon^2 \left\{ \cosh \left(N \cosh^{-1} \left(\frac{\Omega_s}{\Omega_p} \right) \right\}^2 \right]$$

Substituting the value of "ε" in the above equation then

$$\boxed{N \geq \frac{\cosh^{-1} \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1} \left(\dfrac{\Omega_s}{\Omega_p} \right)}} \quad -⑧ \qquad \left[\cosh^{-1} x = \ln \left[x + \sqrt{x^2 - 1} \right] \right]$$

(or)

$$\boxed{N \geq \frac{\cosh^{-1} A}{\cosh^{-1} (1/k)}} \quad -⑨ \quad \text{where} \quad A = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)^{1/2}$$

$$k = \frac{\Omega_p}{\Omega_s}$$

Pole locations for chebyshev Filter.

The poles for the type-I filter are obtained by setting the denominator of eq ① to zero

That is $1+\varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)=0$ — ⑩

put $\Omega_s = \frac{s}{j}$ in the above equation

$$1+\varepsilon^2 C_N^2\left(\frac{s}{j\Omega_p}\right)=0 \Rightarrow 1+\varepsilon^2 C_N^2\left(\frac{s}{j\Omega_p}\times\frac{j}{j}\right)=0.$$

$$1+\varepsilon^2 C_N^2\left(\frac{-js}{\Omega_p}\right)=0 \quad — ⑪$$

$$\varepsilon^2 C_N^2\left(\frac{-js}{\Omega_p}\right)=-1 \Rightarrow C_N^2\left(\frac{-js}{\Omega_p}\right)=\frac{-1}{\varepsilon^2}$$

Taking square root on both sides

$$C_N\left(\frac{-js}{\Omega_p}\right)=\pm\frac{j}{\varepsilon}=\cos\left[N\cos^{-1}\left(\frac{-js}{\Omega_p}\right)\right] \quad — ⑫$$

We define

$$\cos^{-1}\left(\frac{-js}{\Omega_p}\right)=\phi-j\theta \quad — ⑬$$

Substitute the eq ⑬ in eq ⑫ then

$$\pm\frac{j}{\varepsilon}=\cos[N\phi-jN\theta]$$

$$=\cos(N\phi)\cos(jN\theta)+\sin[N\phi]\sin[jN\theta]$$

$$\pm\frac{j}{\varepsilon}=\cos(N\phi)\cosh(N\theta)+j\sin(N\phi)\sinh(N\theta)$$

$$\boxed{\begin{aligned}\cos\theta &= \frac{e^{j\theta}+e^{-j\theta}}{2}\\ \cos j\theta &= \frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2}\\ &= \frac{e^{-\theta}+e^{\theta}}{2}=\cosh\theta\end{aligned}}$$

Equating the real and imaginary parts of both sides of above eq

$$\cos(N\phi)\cosh(N\theta)=0 \quad — ⑭$$

$$\sin(N\phi)\sinh(N\theta)=\pm\frac{1}{\varepsilon} \quad — ⑮$$

Since $\cosh(N\theta) > 0$ for θ real, then in order to satisfy the eq (14)

We have

$$\phi = \frac{(2k-1)\pi}{2N} \quad -(16)$$

Using this result and eq (15) we can now solve for θ, where $\sin N\phi = \pm 1$. Now, We have

$$\theta = \pm \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon}\right) \quad -(17)$$

Combining the eq (16), (17) and eq (13), We obtain the left half plane locations given by

$$S_k = j\Omega_p \cos(\phi - j\theta)$$

$$= j\Omega_p [\cos\phi \cosh\theta + j\sin\phi \sinh\theta]$$

$$= \Omega_p [-\sin\phi \sinh\theta + j\cos\phi \cosh\theta] \quad -(18)$$

The above equation can be simplified using the identity

$$\sinh^{-1}(\varepsilon^{-1}) = \ln(\varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}})$$

(or)

$$\mu = e^{\sinh^{-1}(\varepsilon^{-1})} = \varepsilon^{-1} + \sqrt{1+\varepsilon^{2}}$$

From eq (17) We can write

$$\sinh\theta = \sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{e^{(1/N)\sinh^{-1}(1/\varepsilon)} - e^{-(1/N)\sinh^{-1}(1/\varepsilon)}}{2} \qquad \left[\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right]$$

$$= \frac{\left[e^{\sinh^{-1}(1/\varepsilon)}\right]^{1/N} - \left[e^{-\sinh^{-1}(\varepsilon^{-1})}\right]^{1/N}}{2} \quad -(19)$$

$$\sinh\theta = \frac{\mu^{1/N} - \mu^{-1/N}}{2} \quad -(20)$$

In the same way, $\cosh\theta = \dfrac{\mu^{1/N} + \mu^{-1/N}}{2}$ $\quad -(21)$

Now,

$$S_k = \Omega p\left[-\sin\phi\left(\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right)+j\cos\phi\left(\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right)\right]$$

$$= -a\sin\phi + jb\cos\phi$$

$$= -a\sin\frac{(2k-1)\pi}{2N} + jb\cos\frac{(2k-1)\pi}{2N}$$

$$= a\cos\left[\frac{\pi}{2}+\frac{(2k-1)\pi}{2N}\right] + jb\sin\left[\frac{\pi}{2}+\frac{(2k-1)\pi}{2N}\right]$$

$$S_k = a\cos\phi_k + jb\sin\phi_k \quad \text{— (22)}$$

$S_k = \sigma_k + j\Omega_k$ where $k = 1,2 \ldots N$.

The poles of a chebyshev filter can be determined by using (22)

$$a = \Omega p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] \;;\; b = \Omega p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \quad \text{where } k = 1,2 \ldots N.$$

and $\mu = e^{\sinh^{-1}(\varepsilon^{-1})} = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$

The poles of the chebyshev transfer function are located on an ellipse in the s-plane as shown in fig. The equation of the ellipse is given by

$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1$$

where a and b are minor and major axes of the ellipse respectively

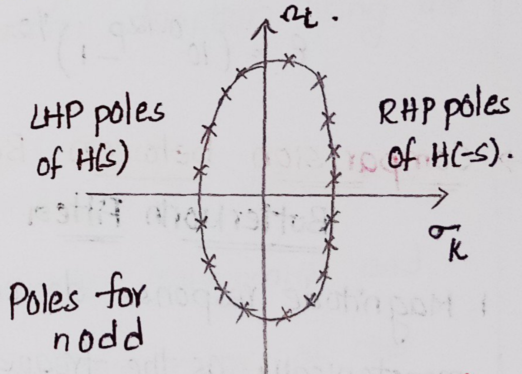


Fig: Locus of the poles of chebyshev filter.

→ Chebyshev Type - 2 Filter:

Chebyshev type-2 filter has poles and zeros. The Magnitude square response is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]}$$

where $C_N(x)$ is the N^{th} order chebyshev polynomial

Ω_s is the stop-band frequency

Ω_p is the pass-band frequency

For the given specifications $\varepsilon, \lambda, \Omega_s$ and Ω_p the order of the filter

$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N = \frac{\cosh^{-1}(A)}{\cosh^{-1}(1/k)}$$

where $A = \frac{\lambda}{\varepsilon}$; $k = \frac{\Omega_p}{\Omega_s}$

$\varepsilon = (10^{0.1\alpha_p}-1)^{1/2}$; $\lambda = (10^{0.1\alpha_s}-1)^{1/2}$

→ Comparision between Butterworth Filter and chebyshev Filter

Butterworth Filter	Chebyshev Filter
1. Magnitude response decreases monotonically as the frequency 'Ω' increases from 0 to ∞	1. Magnitude response exhibits ripples in the passband or stop band according to the type
2. Transition band is more compared to chebyshev filter	2. Transition band is less compared to butterworth filter
3. Poles lie on the circle	3. Poles lie on the ellipse
4. No. of poles in Butterworth are more	4. No. of poles is less than the Butterworth This is a great advantage because less no. of discrete components are required.

→ Steps to design an analog Chebyshev Low Pass filter.

1. From the given specifications find the order of the filter N

2. Round off it to the next higher integer

3. Using the following formulas find the values of 'a' and 'b' which are minor and major axis of the ellipse respectively

$$a = \frac{\Omega p \left[\mu^{1/N} - \mu^{-1/N}\right]}{2} \quad ; \quad b = \Omega p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where $\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$

$\varepsilon = \sqrt{10^{0.1\alpha p} - 1}$

Ωp = Passband frequency

αp = Maximum allowable attenuation in the passband.

4. Calculate the poles of chebyshev filter which lie on an ellipse by using the formula

$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2, \ldots N$

where $\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \qquad k = 1, 2 \ldots N$

5. Find the denominator polynomial of the transfer function using the above poles.

6. The numerator of the transfer function depends on the value of N.

(a) For N odd substitute $s=0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function.

(b) For N even substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

→ Given the specifications $\alpha_p : 3dB$, $\alpha_s : 16dB$, $f_p = 1kHz$ and $f_s = 2kHz$. Determine the order of the filter using chebyshev approximation. Find $H(s)$.

Sol. Given

$\alpha_p = 3dB \qquad f_p = 1kHz$

$\alpha_s = 16dB \qquad f_s = 2kHz$

$\Omega_p = 2\pi f_p = 2\pi \times 1000 = 2000\pi$ rad/sec

$\Omega_s = 2\pi f_s = 2\pi \times 2000\ Hz = 4000\pi$ rad/sec

Step 1 :

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\times16}-1}{10^{0.1\times3}-1}}}{\cosh^{-1}\left(\frac{4000\pi}{2000\pi}\right)}$$

$$= \frac{\cosh^{-1}(6.245)}{\cosh^{-1}(2)}$$

$$= \frac{\ln(6.245+\sqrt{(6.245)^2-1})}{\ln(2+\sqrt{4-1})} \qquad \left[\because \cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})\right]$$

$$N = \frac{2.518}{1.316} = 1.91$$

Step 2 : Round off to next high digit i.e $N = 2$

Step 3 : The values of minor axis and major axis can be found as below

$$\varepsilon = (10^{0.1\alpha_p}-1)^{1/2} = (10^{0.1\times3}-1)^{1/2} = 0.997 = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

$$= 1 + \sqrt{1+1} = 1+\sqrt{2} = 1+1.414 = 2.414$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = \frac{2000\pi}{2}\left[(2.414)^{1/2} - (2.414)^{-1/2}\right]$$

$$= 1000\pi\,[1.553 - 0.643]$$

$$= 909.37\pi = 910\pi$$

$$b = \Omega p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$

$$= \frac{2000\pi}{2}\left[(2.414)^{1/2}+(2.414)^{-1/2}\right] = 2197\pi$$

step 4: The poles are given by

$$S_k = a\cos\phi_k + jb\sin\phi_k \quad ; \quad \phi_k = \frac{\pi}{2}+\frac{(2k-1)\pi}{2N} \quad k = 1,2$$

For $k=1$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1 \qquad \phi_1 = \frac{\pi}{2}+\frac{\pi}{2.2} = \frac{\pi}{2}+\frac{\pi}{4} = \frac{2\pi+\pi}{4} = \frac{3\pi}{4}$$

For $k=2$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2 \qquad \phi_2 = \frac{\pi}{2}+\frac{3\pi}{4} = \frac{2\pi+3\pi}{4} = \frac{5\pi}{4} = 225^0$$

$$\therefore S_1 = a\cos\phi_1 + jb\sin\phi_1$$

$$= 910\pi\left[\cos(135^0)\right] + j2197\pi.(\sin(135))$$

$$= -910\pi \times 0.707 + j1554\pi = -643.37\pi + j1554\pi$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2$$

$$= -643.46\pi - j1554\pi$$

step 5: The denominator of $H(s) = (s+643.46\pi)^2 + (1554\pi)^2$.

step 6: The numerator of $H(s) = \dfrac{(643.46\pi)^2+(1554\pi)^2}{\sqrt{1+\varepsilon^2}} = \dfrac{(2197\pi)^2}{\sqrt{2}}$

$$= \frac{2197 \times 2197\,(\pi)^2}{\sqrt{2}} = (1414.38)^2\pi^2$$

$$\therefore \text{The transfer function } H(s) = \frac{(1414.38)^2\pi^2}{s^2+1287\pi s+(1682)^2\pi^2}$$

→ Determine the order and the poles of a type-I low pass Chebyshev filter that has a 1dB ripple in the passband and passband frequency $\Omega_p = 1000\pi$, a stop band frequency of 2000π and an attenuation of 40dB or more.

Sol. Given data $\alpha_p = 1\,dB$; $\Omega_p = 1000\pi$ rad/sec

$\alpha_s = 40dB$ $\qquad \Omega_s = 2000\pi$ rad/sec

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^4-1}{10^{0.1}-1}}\right)}{\cosh^{-1}\left(\dfrac{2000\pi}{1000\pi}\right)} = 4.536$$

$\therefore N = 5$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1} = \sqrt{10^{0.1\times1}-1} = 0.508$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = \sqrt{(0.508)^{-2}+(1)} + (0.508)^{-1} = 4.17$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 289.5\pi$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 1041\pi$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, 3, 4, 5$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{10} = \frac{5\pi+\pi}{10} = \frac{6\pi}{10} = \frac{3\pi}{5} = 108^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{10} = \frac{5\pi+3\pi}{10} = \frac{8\pi}{10} = \frac{4\pi}{5} = 144^\circ$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{10} = \frac{\pi}{2} + \frac{\pi}{2} = \pi = 180^\circ$$

$$\phi_4 = \frac{\pi}{2} + \frac{7\pi}{10} = \frac{5\pi+7\pi}{10} = \frac{12\pi}{10} = \frac{6\pi}{5} = 216^\circ$$

$$\phi_5 = \frac{\pi}{2} + \frac{9\pi}{10} = \frac{5\pi+9\pi}{10} = \frac{14\pi}{10} = \frac{7\pi}{5} = 252^\circ$$

The poles are $s_k = a\cos\phi_k + jb\sin\phi_k$

$s_1 = -89.5\pi + j989\pi$

$s_2 = -234.2\pi + j612\pi$

$s_3 = -289.5\pi$

$s_4 = -234.2\pi - j612\pi$

$s_5 = -89.5\pi - j989\pi$

→ Frequency Transformation in Analog Domain.

Frequency transformations can be used to design Low pass filters with different pass band frequencies, high pass filters, band pass filters and band stop filters from a normalized low pass analog filter ($\Omega_c = 1$ rad/sec)

1. Lowpass to Lowpass filter.

When a normalized Low pass filter, it is desirable to have a Low pass filter with a different cut-off frequency Ω_c (or Pass band frequency Ω_p). This can be accomplished by the transformation

$$s \rightarrow \frac{s}{\Omega_c}$$

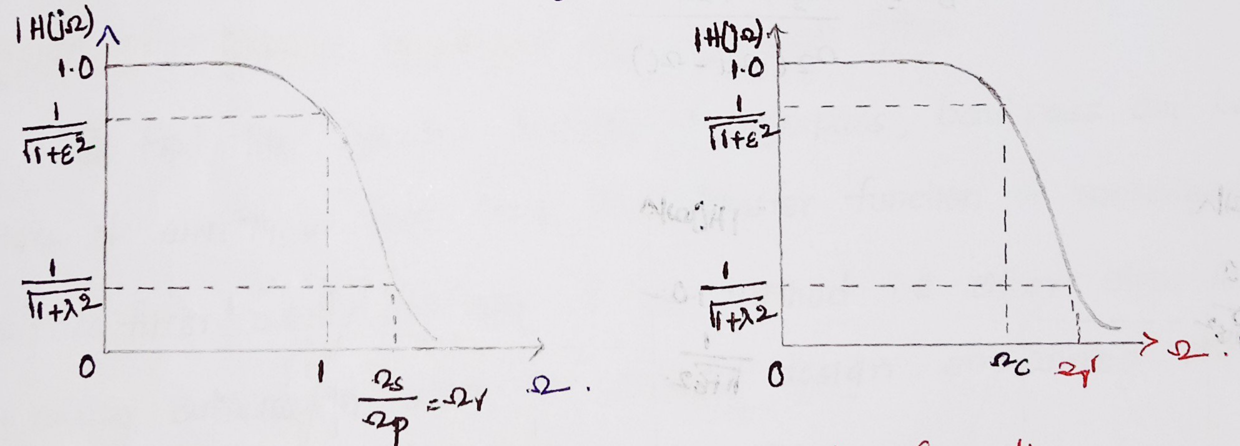


Fig: Lowpass to Lowpass transformation.

2. Lowpass to Highpass

Given a normalized low pass filter, it is desirable to have a high pass filter with cut-off frequency Ω_c. Then the transformation is $s \rightarrow \frac{\Omega_c}{s}$

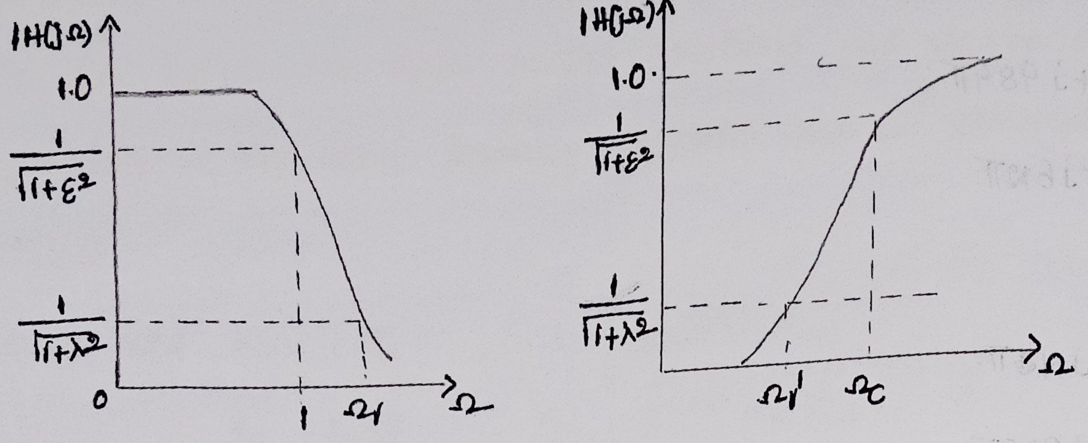


Fig : Lowpass to Highpass transformation.

Lowpass to Bandpass

The transformation for converting a normalized low pass filter to a bandpass filter with cut-off frequencies Ω_l, Ω_u can be accomplished by

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$\Omega_r \Rightarrow \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$$

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

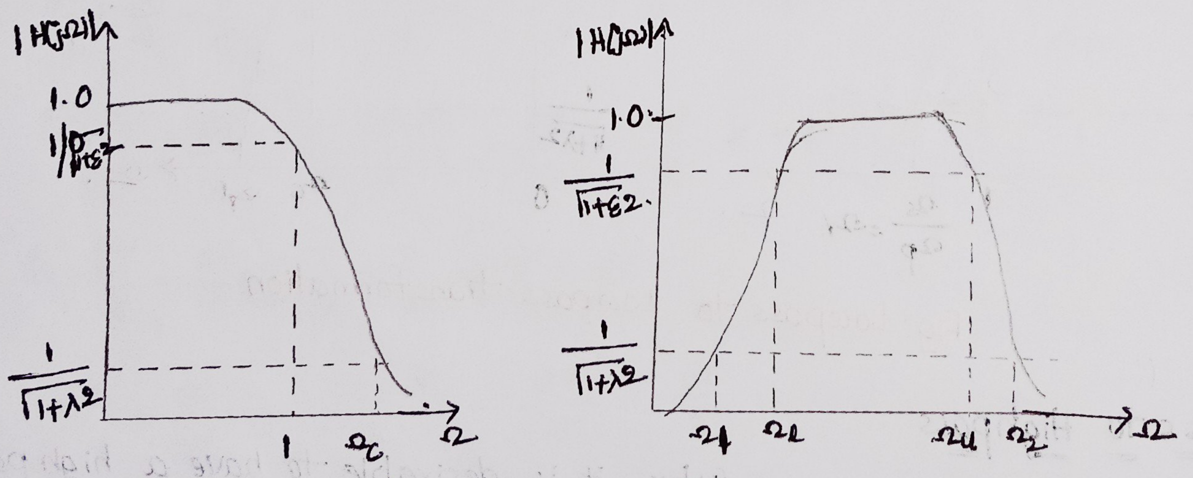


Fig: Lowpass to Bandpass transformation.

Lowpass to Bandstop

The transformation to convert a normalized low pass filter to a band stop filter is

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u}$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u}$$

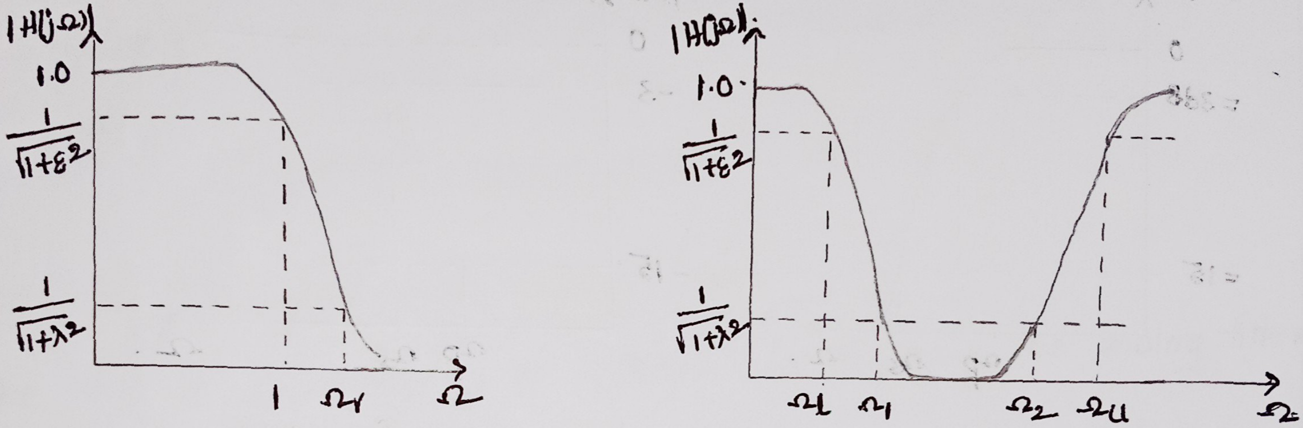


Fig: Lowpass to Bandstop transformation.

→ Design of highpass, bandpass and bandstop filters

To find the transfer function of high pass, band pass and band stop filters of any type first find the transfer function of normalized low pass filter using any one of the method i.e either steps to design an analog Butterworth filter or steps to design an analog chebyshev Lowpass filter.

→ For the given specifications $\alpha_p = 3dB$, $\alpha_s : 15dB$, $\Omega_p : 1000\, rad/sec$, $\Omega_s = 500\, rad/sec$ design a high pass filter.

Sol: First we design a normalized Low pass filter and then use suitable transformation to get the transfer function of a high pass filter.

For low pass filter

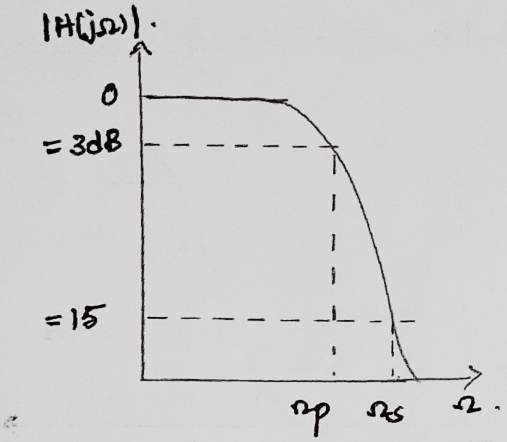
$\Omega_c = \Omega_p = 500\, rad/sec$

$\Omega_s = 1000\, rad/sec.$

For high pass filter

$\Omega_c = \Omega_p = 1000\, rad/sec$

$\Omega_s = 500\, rad/sec$



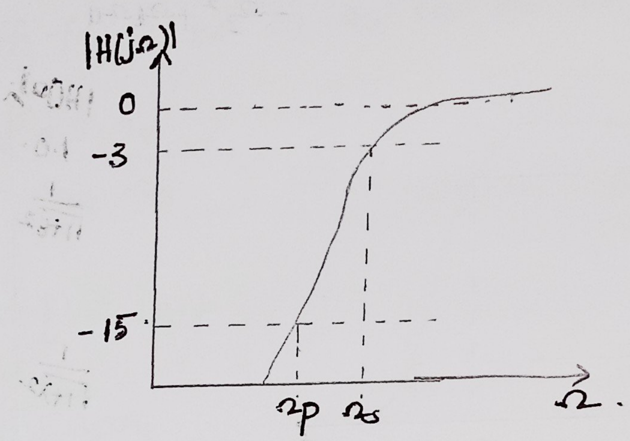


Fig: Low pass to High pass transformation

Low pass filter specifications

$\Omega_c = \Omega_p = 500\, rad/sec$; $\alpha_p : 3dB$

$\Omega_s : 1000\, rad/sec$; $\alpha_s : 15dB$

We know that
$$N \geq \frac{\log\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)^{1/2}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{10^{0.1\times15}-1}{10^{0.1\times3}-1}\right)^{1/2}}{\log\left(\frac{15}{3}\right)}$$

$$N \geq \frac{\log(5.533)}{\log(5)} = 2.468.$$

$\therefore N = 3$

The transfer function $H(s)$ for $\Omega_c = 1$ rad/sec and $N=3$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

To get high pass filter having cut-off frequency

$$\Omega_c = \Omega_p = 1000 \text{ rad/sec}$$

substitute $s \to \frac{1000}{s}$

$$\therefore H_a(s) = H(s)\Big|_{s \to \frac{1000}{s}}$$

$$= \frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s \to \frac{1000}{s}}$$

$$H_a(s) = \frac{s^3}{(s+1000)(s^2+1000s+(1000)^2)}$$

→ Design of IIR filters from analog filters

The four most widely used methods for digitizing the analog filter into a digital filter include

1. Approximation of derivatives
2. The impulse invarient transformation
3. The Bilinear transformation
4. The matched z-transformation technique

If the conversion technique is to be effective, it should possess the following desirable properties.

1. The $j\Omega$- axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.
2. The left-half plane of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

→ Steps to design a digital filter using Impulse Invariance Method

1. For the given specifications, find $H_a(s)$, the transfer function of an analog filter.

2. Select the sampling rate of the digital filter, T seconds per sample.

3. Express the analog filter transfer function as the sum of single-pole filters.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$$

4. Compute the z-transform of the digital filter by using the formula

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

For high sampling rates use

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{P_k T} z^{-1}}$$

→ For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Sol. Given $H(s) = \frac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = (s+1) \cdot H(s) \Big|_{s=-1} = \frac{\cancel{(s+1)} \cdot 2}{\cancel{(s+1)}(s+2)} \Big|_{s=-1} = \frac{2}{1} = 2$$

$$B = (s+2) \cdot H(s) \Big|_{s=-2} = \frac{\cancel{(s+2)} \cdot 2}{(s+1)\cancel{(s+2)}} \Big|_{s=-2} = \frac{2}{-2+1} = -2$$

$\therefore H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$

$= \frac{2}{s-(-1)} - \frac{2}{(s-(-2))}$

using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_kT}z^{-1}}$$

i.e $(s-P_k)$ is transformed to $1-e^{P_kT}z^{-1}$

There are two poles $P_1 = -1$ & $P_2 = -2$. so

$$H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$$

For T = 1sec

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}.$$

→ Design a third order Butterworth digital filter using impulse invarient technique. Assume sampling period T = 1sec.

Sol: We know that the transfer function of Butterworth filter with order N=3

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s+0.5+j0.866)} + \frac{C}{(s+0.5-j0.866)}$$

$$A = (s+1)\frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s=-1} = \frac{1}{1-1+1} = 1$$

$$B = (s+0.5+j0.866)\cdot\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\Bigg|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-0.5-j0.866+0.5-j0.866)}$$

$$= \frac{1}{(0.5-j0.866)(-j1.732)} = \frac{1}{-1.5-j0.866}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$H(s) = \frac{1}{(s+1)} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5+j0.866)}$$

In Impulse Invariant technique

if $H(s) = \sum_{k=1}^{N}\frac{c_k}{s-P_k}$ then $H(z) = \sum_{k=1}^{N}\frac{c_k}{1-e^{P_kT}z^{-1}}$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} + \frac{-0.5+0.288j}{1-e^{(-0.5-j0.866)T}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5+j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

→ Design of IIR filter using Bilinear Transformation

Steps to design digital filter using Bilinear Transform technique

1. From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

2. Using the analog frequencies find $H(s)$ of the analog filter.
3. Select the sampling rate of the digital filter, call it 'T' seconds per sample
4. Substitute $s = \frac{2}{T}\frac{(1-z^{-1})}{1+z^{-1}}$ into the transfer function found in step 2.

→ Apply Bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$ sec find $H(z)$.

Sol. Given $H(s) = \frac{2}{(s+1)(s+2)}$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{(s+1)(s+2)}\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{\{2(1-z^{-1})+(1+z^{-1})\}\{2(1-z^{-1})+2(1+z^{-1})\}}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})(2-2z^{-1}+2+2z^{-1})} = \frac{\cancel{2}(1+z^{-1})^2}{(3-z^{-1})(\cancel{4})_{2}} = \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

→ The Warping effect

Let Ω and ω represent the frequency variables in the analog filter and the derived digital filter respectively. We know the eq

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

For small value of ω

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

$$\omega = \Omega T$$

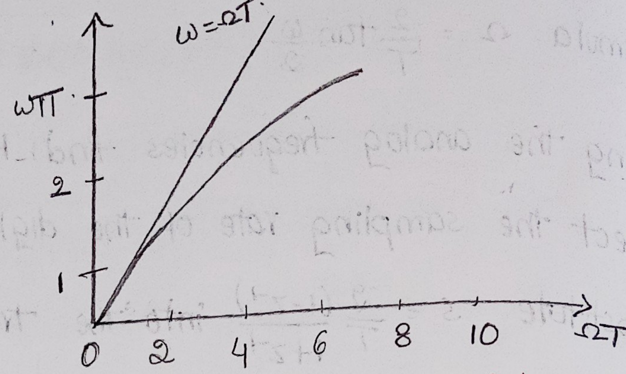


Fig: Relationship between 'Ω' and 'ω'

For low frequencies the relationship between Ω and ω are linear, as a result the digital filter have the same amplitude response as the analog filter.

For high frequencies, however the relationship between ω and Ω becomes non-linear and distortion is introduced in the frequency scale of the digital filter to that of the analog filter. This is known as the "Warping effect" as shown in fig

The influence of the warping effect on the amplitude response is shown in below fig by considering an analog filter with a no. of passbands centered at regular intervals. The derived digital filter will have same no: of passbands. But the center frequencies and bandwidth of higher frequency passband will tend to reduce disproportionately.

The influence of the warping effect on the phase response is shown in fig. Considering an analog filter with linear phase response, the phase response of the derived digital filter will be non-linear.

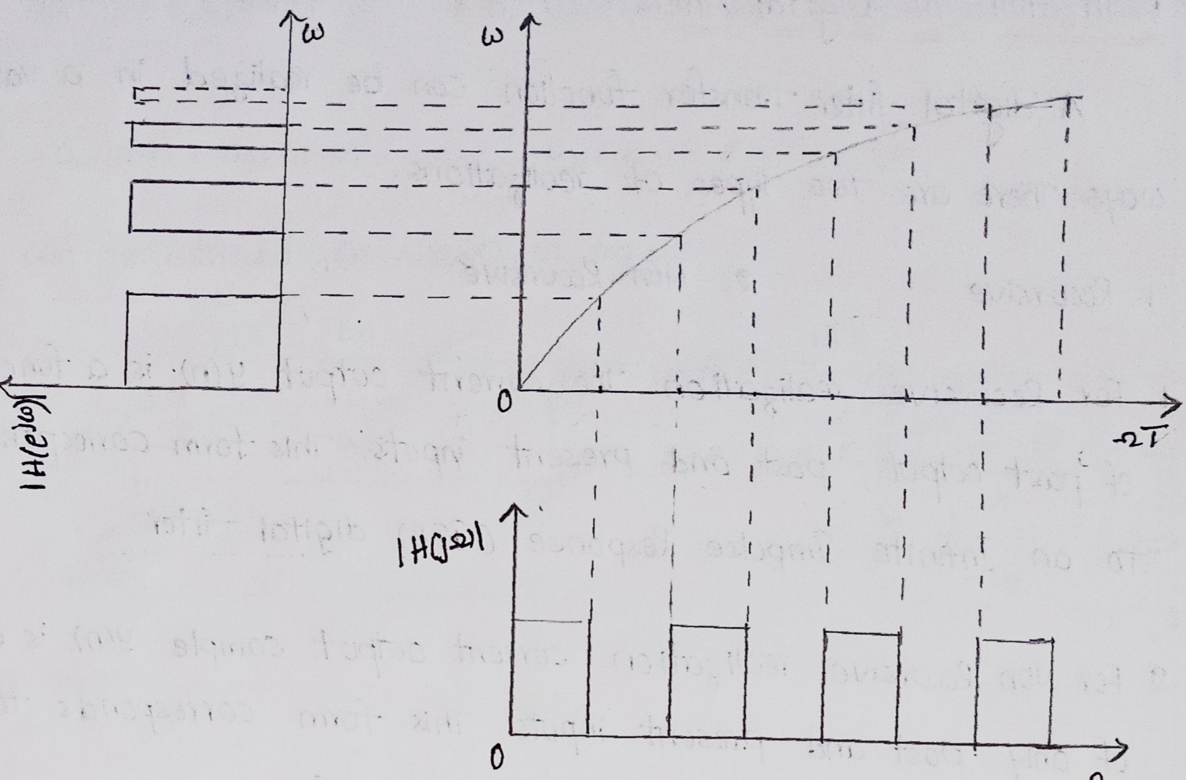


Fig: The effect on magnitude response due to warping effect.

→ Prewarping

The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Therefore, we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

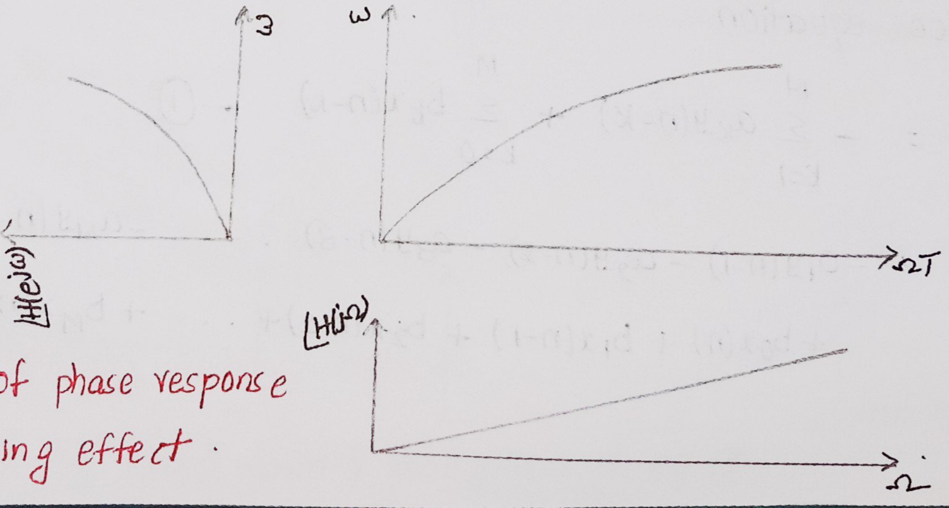


Fig: The effect of phase response due to warping effect.

Realization of Digital Filters

A digital filter transfer function can be realized in a variety of ways. There are two types of realizations.

1. Recursive   2. Non-Recursive

1. For Recursive realization the current output y(n) is a function of past output, past and present inputs. This form corresponds to an Infinite Impulse Response (IIR) digital filter.

2. For Non-Recursive realization current output sample y(n) is a function of only past and present inputs. This form corresponds to Finite Impulse Response (FIR) digital filter.

IIR filters can be realized in many forms. They are

1. Direct form-I realization
2. Direct form-II realization
3. Transposed direct form realization
4. Cascade form realization
5. Parallel form realization
6. Lattice - ladder structure

Direct Form-I realization

Let us consider an LTI recursive system described by the difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \text{—} ①$$

$$= -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) \dots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_M x(n-M) \quad \text{—} ②$$

Let $b_0x(n) + b_1x(n-1) + \dots + b_Mx(n-M) = \omega(n)$ — ③

Then
$$y(n) = -a_1y(n-1) - a_2y(n-2) + \dots - a_Ny(n-N) + \omega(n) \quad \text{— ④}$$

The eq ③ can be realized as shown in fig 1

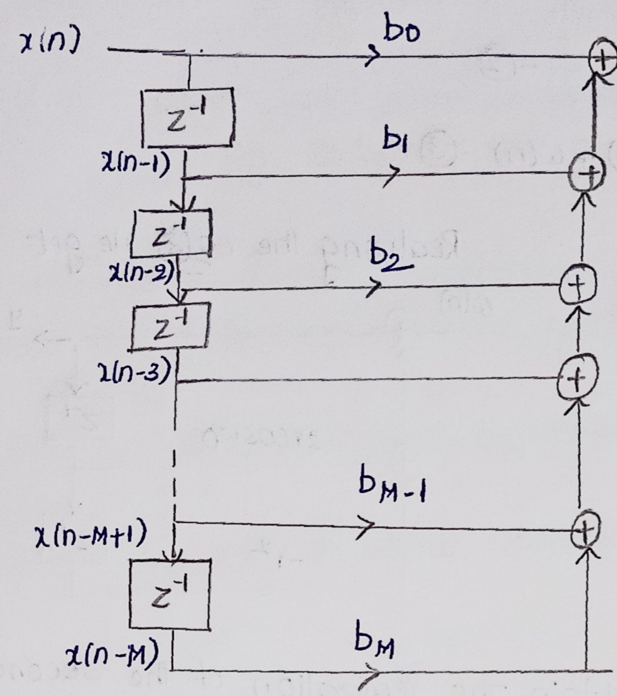


Fig: Realization structure of eq ③

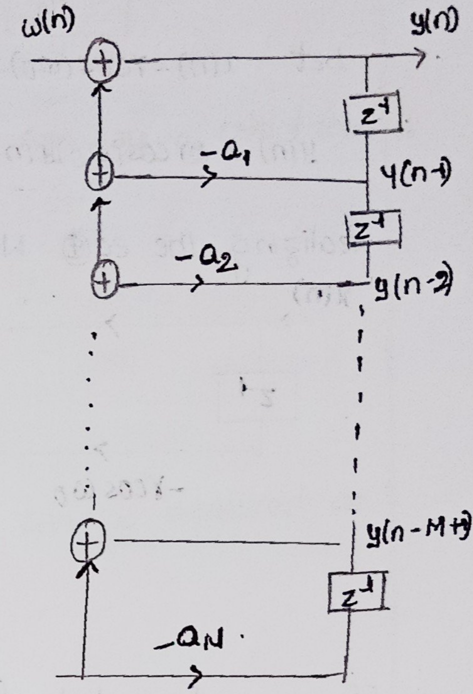


Fig2: Realization of eq ④

Similarily the eq ④ can be realized in Fig 2

To realize the difference eq ② combine Fig 1 and Fig 2

The structure shown in fig3 is called direct form-I which used separate delays for both input and output. This realization requires M+N+1 multiplications, M+N additions and M+N+1 memory locations.

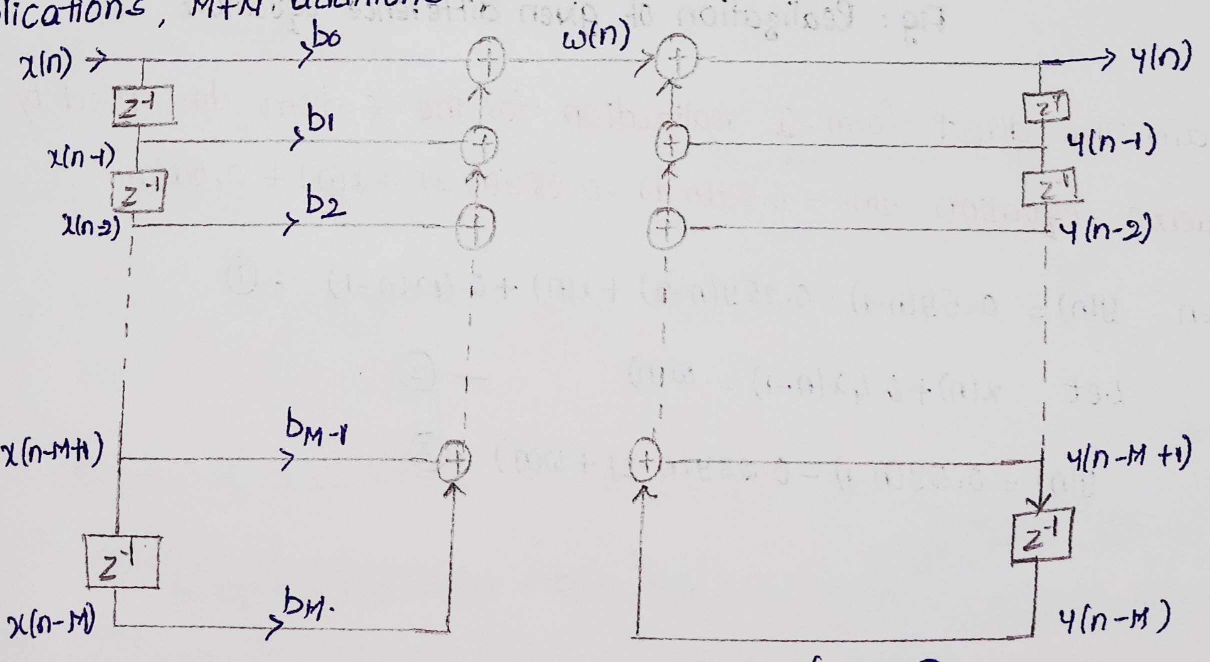


Fig: Direct form I Realization of eq ②

→ Realize the second order digital filter $y(n) = 2r\cos(\omega_0)y(n-1) - r^2y(n-2) + x(n) - r\cos(\omega_0)x(n-1)$

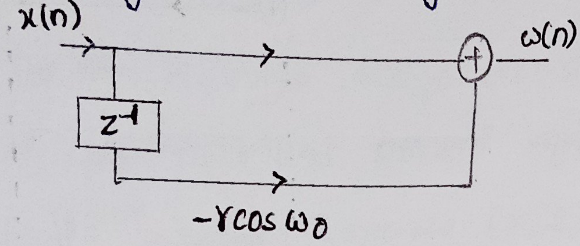
Sol. Given difference equation is

$$y(n) = 2r\cos(\omega_0)y(n-1) - r^2y(n-2) + x(n) - r\cos(\omega_0)x(n-1) \quad -①$$

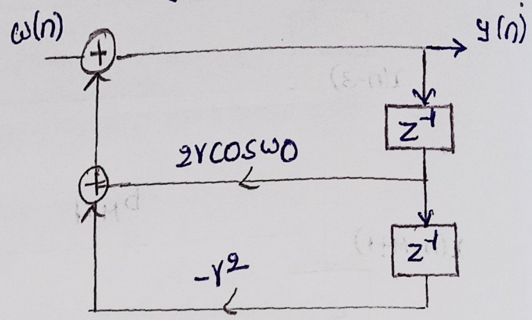
Let $x(n) - r\cos(\omega_0)x(n-1) = \omega(n) \quad -②$

$$y(n) = 2r\cos(\omega_0)y(n-1) - r^2y(n-2) + \omega(n) \quad -③$$

Realizing the eq②. We get



Realizing the eq③ we get



If we combine both figures, we obtain the realization of the second order digital filter as shown in fig3

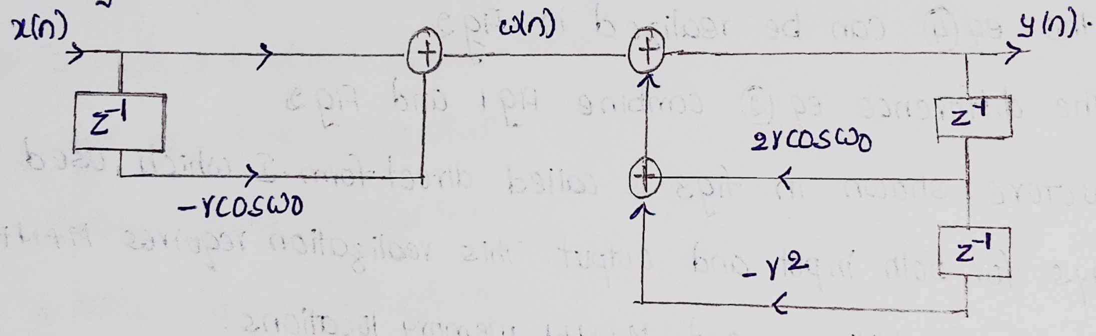


Fig: Realization of given difference equation

→ Obtain the direct form - I realization for the system described by difference equation $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$.

Sol. Given $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1) \quad -①$

Let $x(n) + 0.4x(n-1) = \omega(n) \quad -②$

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + \omega(n) \quad -③$$

Realizing the eq ②, we get

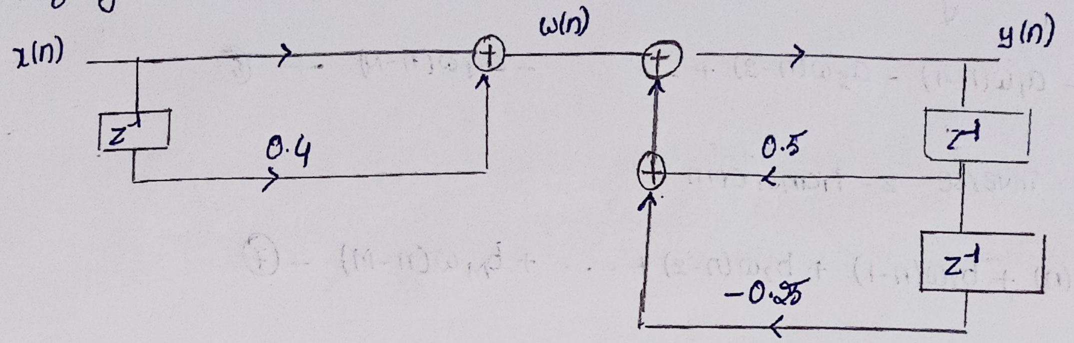


Fig: Realization of given difference equation in Direct form I

→ Direct form II Realization

Consider the difference equation of the form

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

The system function of above difference equation can be expressed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad — ①$$

$$\text{Let } \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} \quad — ②$$

$$\text{where } \frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad — ③$$

$$W(z) + \sum_{k=1}^{N} a_k z^{-k} W(z) = X(z)$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \cdots - a_N z^{-n} W(z) \quad — ④$$

$$\text{and } \frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = W(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$= b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z) \quad — ⑤$$

Now eq ④, taking inverse z-transform

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) + \dots - a_N w(n-N) \quad — ⑥$$

Eq ⑤, taking inverse z-transform

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \dots + b_M w(n-M) \quad — ⑦$$

From eq ⑥ & ⑦ we observe that the same delay terms $w(n-1), w(n-2) \dots$ are used to express $w(n)$ and $y(n)$

The realization of eq ⑥ and ⑦ are shown in below figure

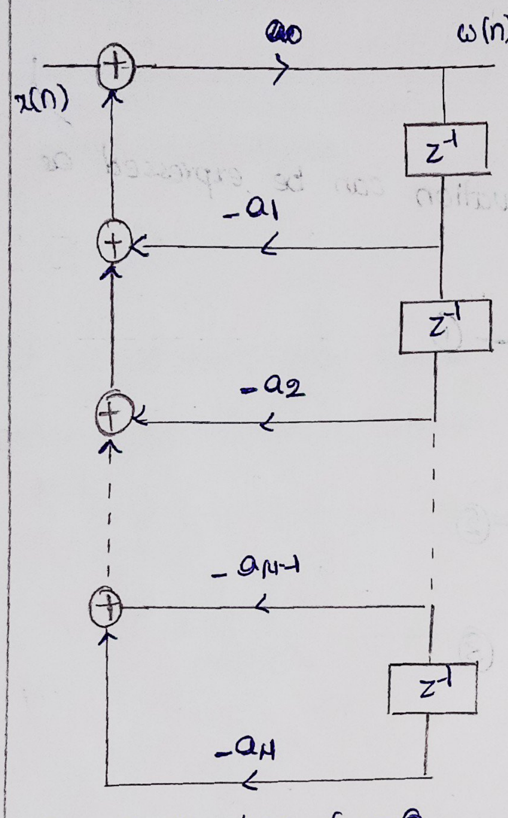


Fig: Realization of eq ⑥

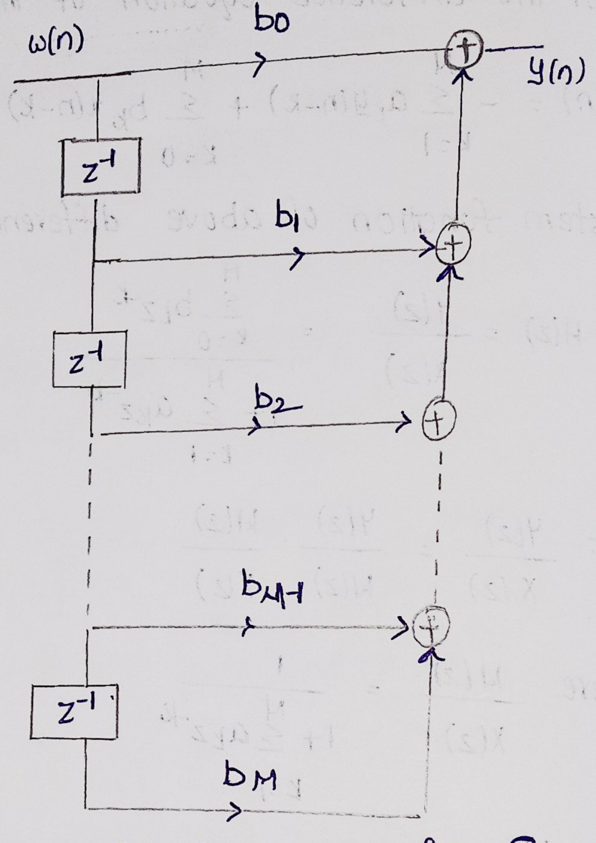


Fig: Realization of eq ⑦

From the above figure, we find that the two delay elements contain the same input $w(n)$ and hence the same output $w(n-1)$. Consequently we can merge these delays into one delay and redraw the figure

The realization structure shows the direct form-II realization. This structure requires $M+N+1$ multiplications, $M+N$ additions and the maximum of (M, N) memory locations. since direct form-II realization minimizes the no. of memory locations, it is called "Canonic"

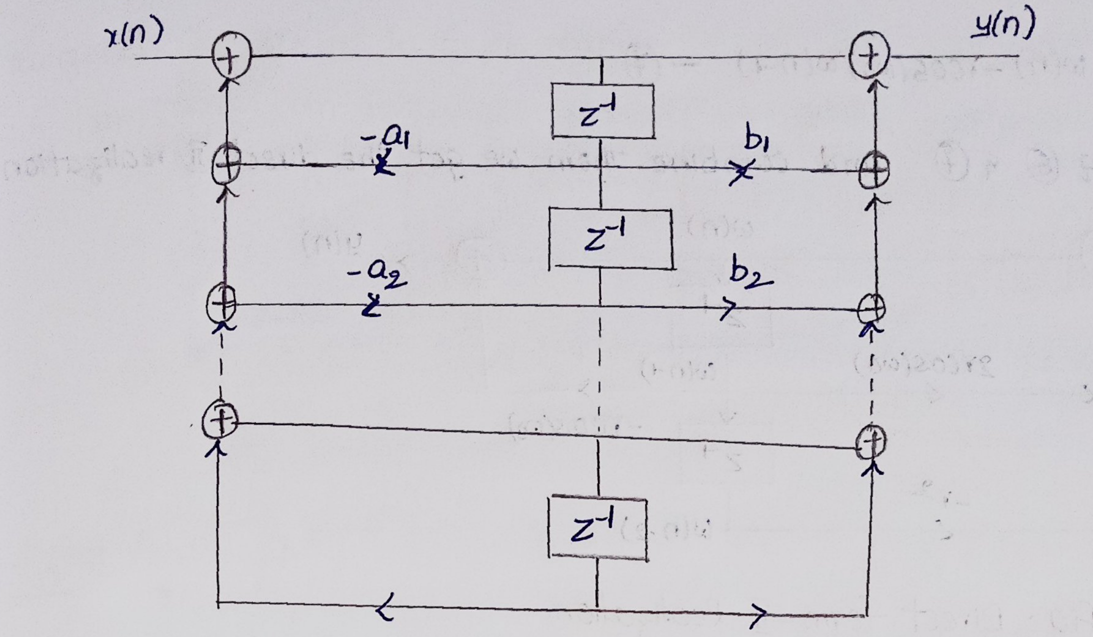


Fig: Direct form-II realization

→ Realize the second order system $y(n) = 2r\cos(\omega_0)y(n-1) - r^2 y(n-2) + x(n) - r\cos(\omega_0)x(n-1)$ in direct form-II.

Sol: Given $y(n) = 2r\cos(\omega_0)y(n-1) - r^2y(n-2) + x(n) - r\cos(\omega_0)x(n-1)$ — ①

Apply z-transform on both sides

$$Y(z) = 2r\cos(\omega_0)z^{-1}Y(z) - r^2z^{-2}Y(z) + X(z) - rz^{-1}\cos(\omega_0)X(z) \quad \text{— ②}$$

$$Y(z)\left[1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}\right] = X(z)\left[1 - rz^{-1}\cos(\omega_0)\right]$$

$$\frac{Y(z)}{X(z)} = \cancel{\frac{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}{1+rz^{-1}\cos(\omega_0)}} = \frac{1 - rz^{-1}\cos(\omega_0)}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}} \quad \text{— ③}$$

Let $\dfrac{Y(z)}{X(z)} = \dfrac{Y(z)}{W(z)} \cdot \dfrac{W(z)}{X(z)}$

where $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$ — ④ ; $\dfrac{Y(z)}{W(z)} = 1 - rz^{-1}\cos(\omega_0)$ — ⑤

By taking inverse z-transform to eq ④, we get

$$w(n) - 2r\cos(\omega_0)w(n-1) + r^2w(n-2) = x(n)$$

$$w(n) = x(n) + 2r\cos(\omega_0)w(n-1) - r^2w(n-2) \quad \text{— ⑥}$$

By taking inverse z-transform to eq (5), We get

$$y(n) = w(n) - r\cos(\omega_0)\, w(n-1) \quad -(7)$$

Realize eq (6) & (7) and combine them we get the direct-II realization

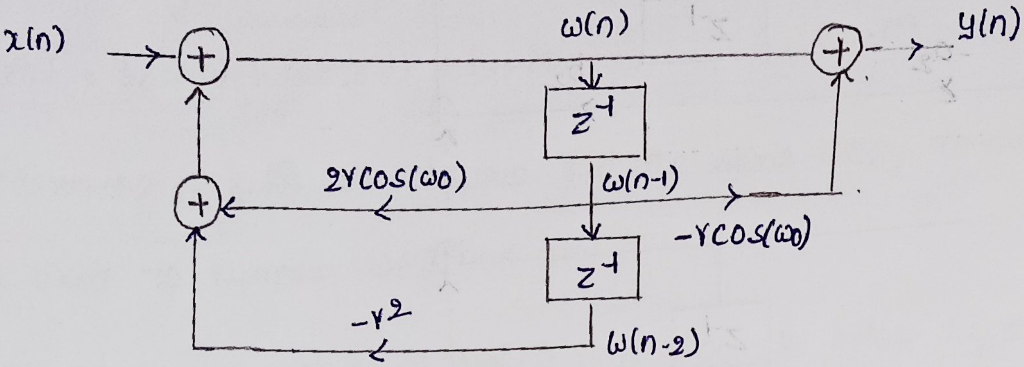


Fig: Direct form-II Realization

→ Determine the direct form-II realization for the following system

$y(n) = 0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$

sol. Given $y(n) = 0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$

$$Y(z) = 0.1z^{-1}Y(z) + 0.72z^{-2}Y(z) + 0.7X(z) - 0.252z^{-2}X(z) \quad -(1)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right] = X(z)\left[0.7 - 0.25z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} \quad -(2)$$

Let $\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2} \quad -(3)$

$$Y(z) = 0.7W(z) - 0.252z^{-2}W(z)$$

$$y(n) = 0.7w(n) - 0.252w(n-2) \quad -(4)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$X(z) = W(z) + 0.1z^{-1}W(z) - 0.72z^{-2}W(z)$$

$$x(n) = w(n) + 0.1\cancel{z^{-1}}w(n-1) - 0.72w(n-2)$$

$$w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2) \quad -(5)$$

The equation ④ & ⑤ are realize and combine then we get direct form-II realization as shown in below figure

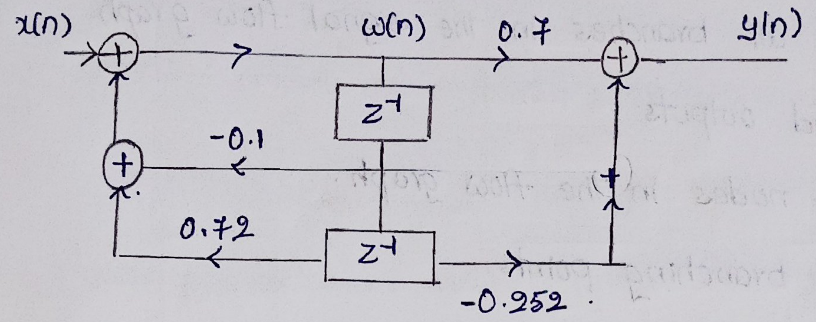


Fig: Direct form-II realization.

→ Signal flowgraph

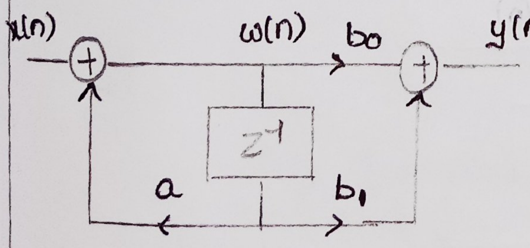
A signal flowgraph is a graphical representation of the relationship between the variables of a set of linear difference equations. The basic elements of a signal flow graph are branches and nodes.

A node represents a system variable, which is equal to the sum of incoming signals from all branches connecting to the node. There are two types of nodes. source node and sink node.

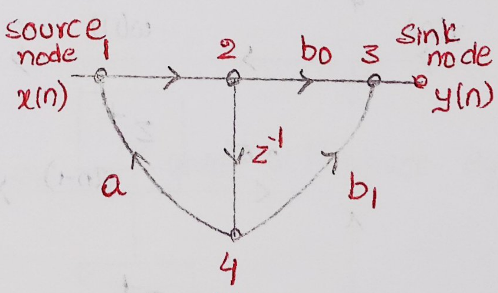
Source nodes are nodes that have no entering branches. Sink nodes are nodes that have only entering branches.

The delay is indicated by the branch transmittance "z^{-1}".

Let us consider a block diagram representation of a first order digital filter shown in figure. The system block diagram can be converted to the signal flow graph as shown in fig



(a)



(b)

Fig(a): Block diagram representation of first-order digital filter

(b): signal flow graph representation of first-order digital filter

→ Transposition theorem and Transposed structure

The transpose of a structure is defined by the following operations.

1. Reverse the direction of all branches in the signal flow graph.
2. Interchange the inputs and outputs
3. Reverse the roles of all nodes in the flow graph.
4. Summing points become branching points.
5. Branching points become summing points.

→ Determine the direct form-II and Transposed direct form II for the given system $y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$

Sol. Given $y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$

Taking z-transform on both sides

$$Y(z) = \frac{1}{2}z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right] = X(z)\left[1+z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}.$$

Let $\frac{W(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$ $\qquad$ $\frac{Y(z)}{W(z)} = 1+z^{-1}.$

$W(z)\left[1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right] = X(z)$ $\qquad$ $Y(z) = (1+z^{-1})W(z)$

$w(n) - \frac{1}{2}w(n-1) + \frac{1}{4}w(n-2) = x(n)$ $\qquad$ $y(n) = w(n) + w(n-1)$

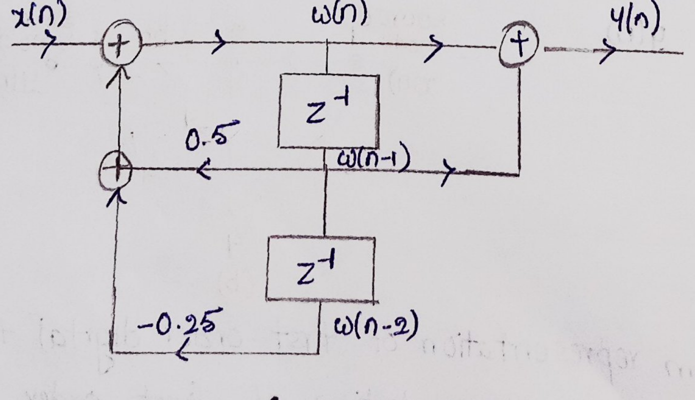


Fig: Direct form-II realization

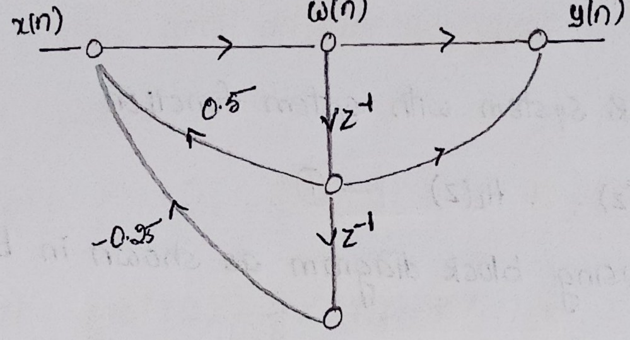


Fig: signal flow graph representation

To get transposed direct form-II do the following operations.

1. change the direction of all branches
2. Interchange the input and output
3. change the summing point to branching point and vice-versa

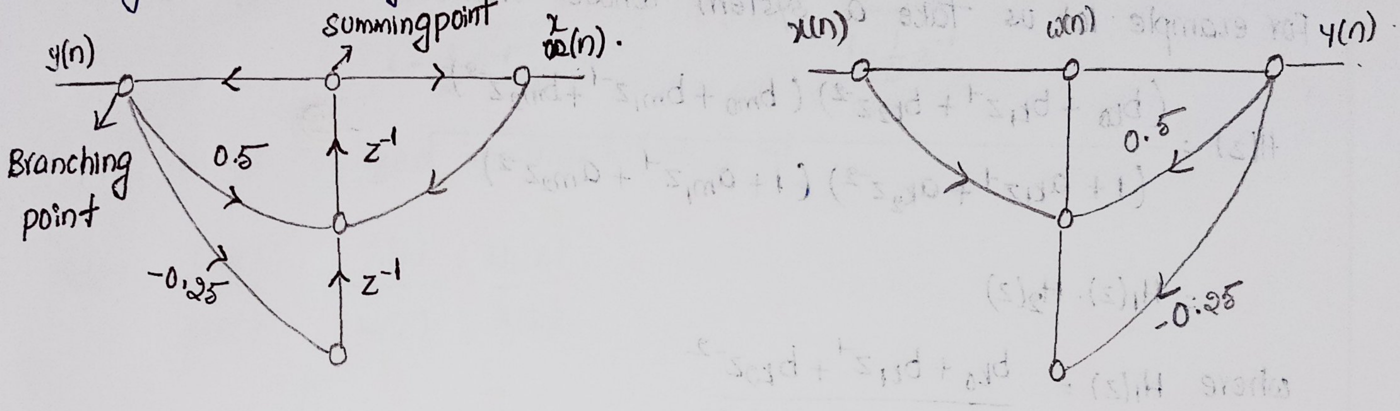


Fig: steps of operations in transposition

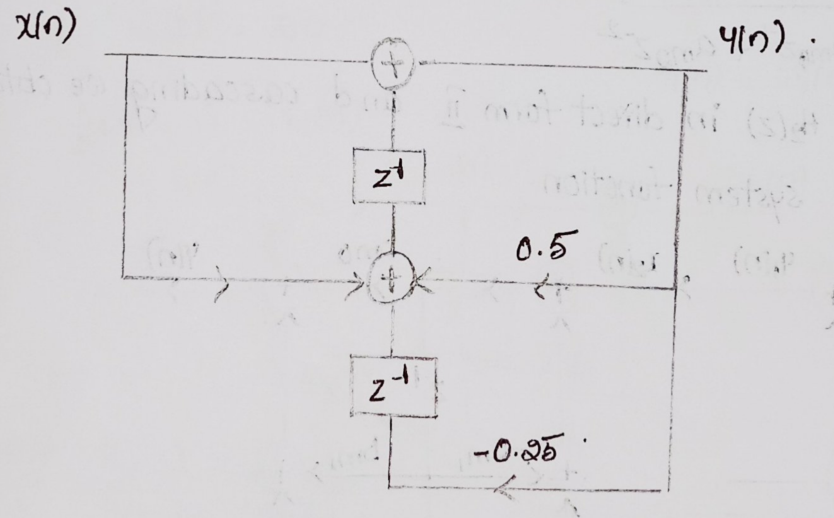


Fig: Transposition structure of given difference equation.

Cascade Form

Let us consider an IIR system with system function

$$H(z) = H_1(z)\, H_2(z) \ldots H_k(z) \quad -① $$

This can be represented using block diagram as shown in below figure

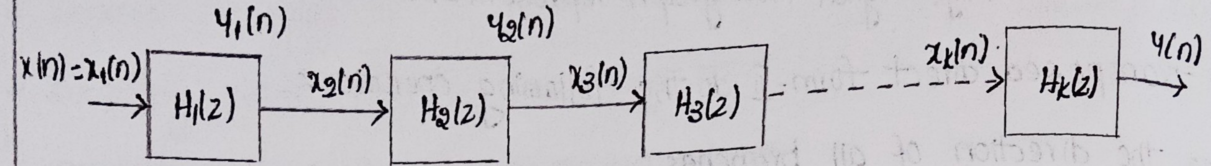


Fig: Block diagram representation of eq ①

Now realize each $H_k(z)$ in direct form II and cascade all structures.

For example let us take a system whose transfer function

$$H(z) = \frac{(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})\,(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})\,(1 + a_{m1}z^{-1} + a_{m2}z^{-2})} \quad -②$$

$$= H_1(z) \cdot H_2(z)$$

where $H_1(z) = \dfrac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$

$$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$$

Realizing $H_1(z)$ and $H_2(z)$ in direct form II and cascading we obtain cascade form of the system function.

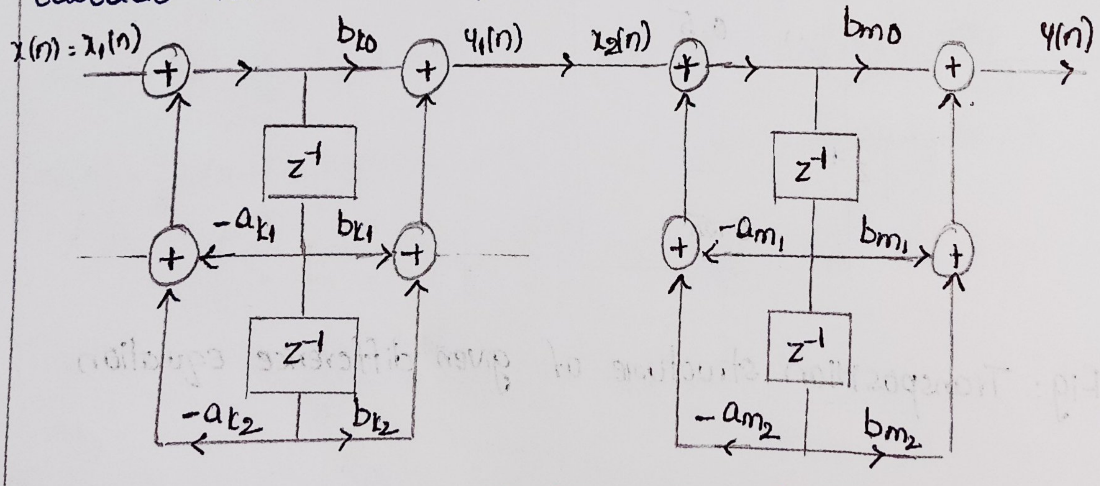


Fig: Cascade Realization of eq ②

→ Realize the system with difference equation $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$ in cascade form.

Sol: Given $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1 + \frac{1}{3}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = H_1(z) \cdot H_2(z)$$

where $H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$; $H_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$

$H_1(z)$ can be realized in direct form II as

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X_1(z)}$$

Let $\frac{W_1(z)}{X_1(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$

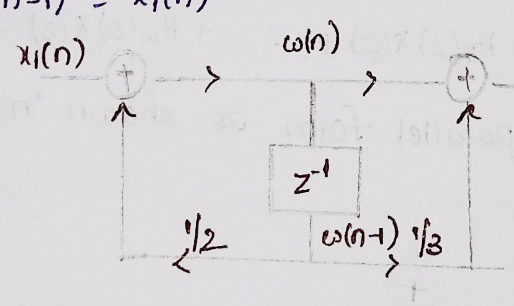
$$W_1(z) - \frac{1}{2}z^{-1}W_1(z) = X_1(z)$$

$$\omega_1(n) - \frac{1}{2}\omega_1(n-1) = x_1(n)$$

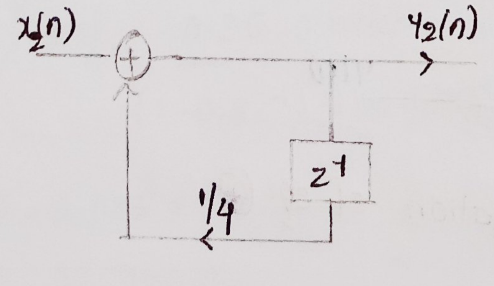
Let $\frac{Y_1(z)}{W_1(z)} = 1 + \frac{1}{3}z^{-1}$

$$Y_1(z) = W_1(z) + \frac{1}{3}z^{-1}W_1(z)$$

$$y_1(n) = \omega_1(n) + \frac{1}{3}\omega_1(n-1)$$



Similarily $H_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$



Cascading the realization of $H_1(z)$ and $H_2(z)$ we have

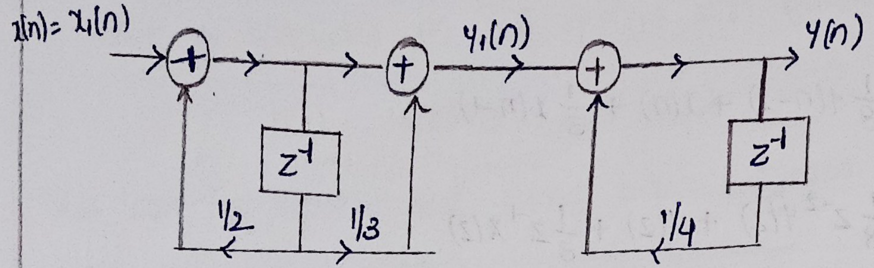


Fig: Cascade realization of given difference equation

→ **Parallel form structure**

In parallel form realization of an IIR system can be obtained by performing a partial expansion of

$$H(z) = C + \sum_{k=1}^{N} \frac{C_k}{1 - P_k z^{-1}} \quad -(1)$$

where $\{P_k\}$ are the poles.

The eq (1) can be written as

$$H(z) = C + \frac{C_1}{1-P_1 z^{-1}} + \frac{C_2}{1-P_2 z^{-1}} + \cdots + \frac{C_N}{1-P_N z^{-1}} \quad -(2)$$

$$H(z) = \frac{Y(z)}{X(z)} = C + H_1(z) + H_2(z) + \cdots + H_N(z) \quad -(3)$$

$$Y(z) = CX(z) + H_1(z)X(z) + H_2(z)X(z) + \cdots + H_N(z)X(z) \quad -(4)$$

The eq (4) can be realized in parallel form as shown in below figure:

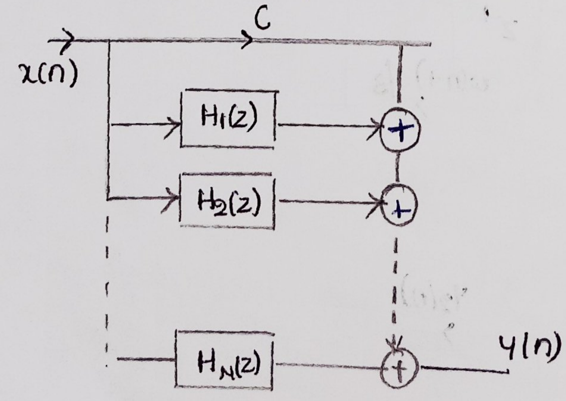


Fig: Parallel form realization of eq (4).

P.S → Realize the system given by difference equation $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$ in parallel form.

Sol. The system function of the difference equation is

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$$

By taking z-transform on both sides

$$Y(z) = -0.1z^{-1}Y(z) + 0.72z^{-2}Y(z) + 0.7X(z) - 0.252z^{-2}X(z)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right] = X(z)\left[0.7 - 0.252z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}.$$

$$-0.72z^{-2} + 0.1z^{-1} + 1 \overline{)\, -0.252z^{-2} + 0.7 \,}\ 0.35$$
$$-0.252z^{-2} + 0.035z^{-1} + 0.35$$
$$+ \quad - \quad -$$
$$-0.035z^{-1} + 0.35$$

$$= 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$H(z) = 0.35 + \frac{0.35 - 0.035z^{-1}}{(1 - 0.8z^{-1})(1 + 0.9z^{-1})}$$

Let $$\frac{0.35 - 0.035z^{-1}}{(1 - 0.8z^{-1})(1 + 0.9z^{-1})} = \frac{A}{(1 - 0.8z^{-1})} + \frac{B}{(1 + 0.9z^{-1})}.$$

$$A = \cancel{(1 - 0.8z^{-1})} \cdot \frac{0.35 - 0.035z^{-1}}{\cancel{(1 - 0.8z^{-1})}(1 + 0.9z^{-1})}\Bigg|_{z^{-1} = \frac{1}{0.8}}$$

$$= \frac{0.35 - 0.035 \times \frac{1}{0.8}}{1 + 0.9 \times \frac{1}{0.8}} = \frac{0.35 - 0.043}{1 + 1.125} = \frac{0.307}{2.125} = 0.144$$

$$B = \cancel{(1 + 0.9z^{-1})} \cdot \frac{0.35 - 0.035z^{-1}}{(1 - 0.8z^{-1})\cancel{(1 + 0.9z^{-1})}}\Bigg|_{z^{-1} = \frac{-1}{0.9}}$$

$$= \frac{0.35 - 0.035 \times \left(\frac{-1}{0.9}\right)}{1 + 0.8 \times \frac{1}{0.9}} = \frac{0.35 + 0.038}{1 + 0.888} = \frac{0.388}{1.888} = 0.206$$

$$\therefore H(z) = 0.35 + \frac{0.206}{1+0.9z^{-1}} + \frac{0.144}{1-0.8z^{-1}}$$

$$= C + H_1(z) + H_2(z)$$

$H_1(z)$ can be realized in direct form II as

$$H_1(z) = \frac{0.206}{1+0.9z^{-1}} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X_1(z)}$$

Let $\frac{W_1(z)}{X_1(z)} = \frac{1}{1+0.9z^{-1}}$

$$X_1(z) = W_1(z) + 0.9z^{-1}W_1(z)$$

By taking inverse z-transform

$$x_1(n) = w_1(n) + 0.9w_1(n-1)$$
$$w_1(n) = x_1(n) - 0.9w_1(n-1)$$

Let $\frac{Y_1(z)}{W_1(z)} = 0.206$

$$Y_1(z) = W_1(z)\,0.206$$

By taking inverse z-transform

$$y_1(n) = w_1(n)\,0.206$$

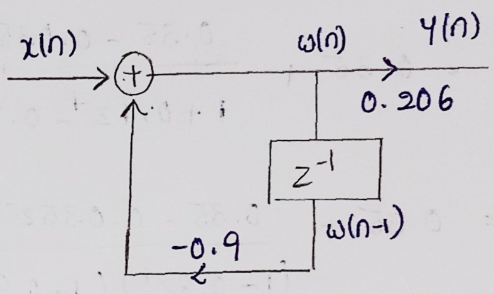


Fig: $H_1(z)$ realization

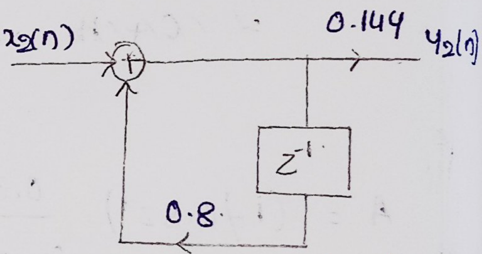


Fig: $H_2(z)$ realization

Similarily $H_2(z)$ can be realized in directform-II

Now, the realization of $H(z)$ is shown in fig below

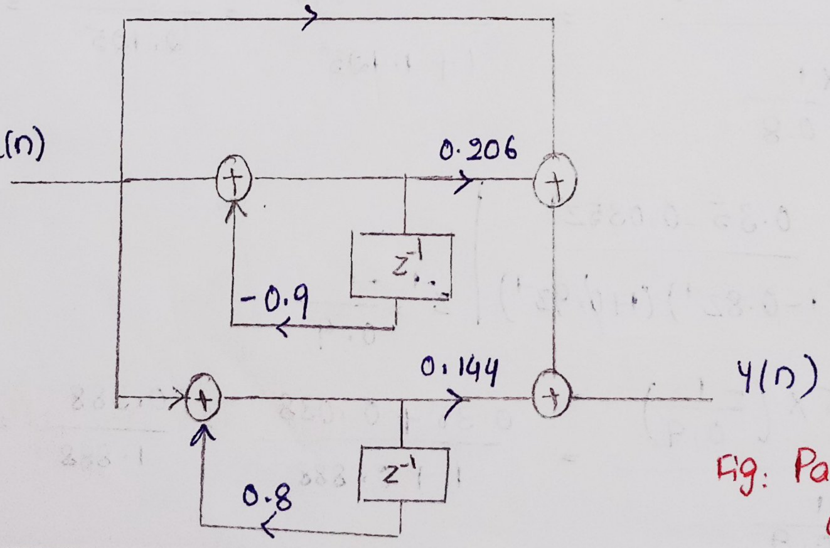


Fig: Parallel form realization of given example.

→ obtain the direct form I, direct form II, cascade and parallel form realization for the system $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

Sol. **Direct form - I**

Let $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3(x(n)) + 3.6x(n-1) + 0.6x(n-2)$

$3x(n) + 3.6x(n-1) + 0.6x(n-2) = w(n)$

$y(n) = -0.1y(n-1) + 0.2y(n-2) + w(n)$

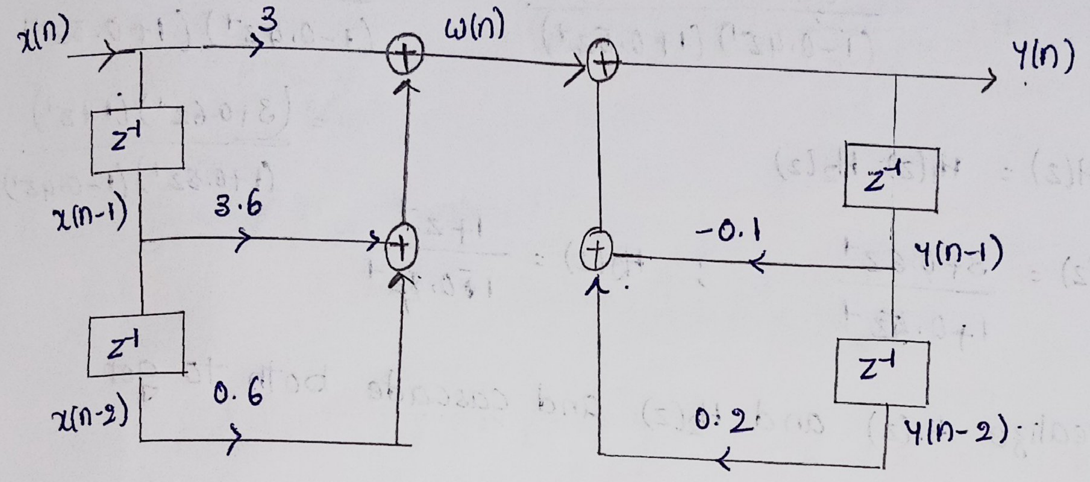


Fig: Direct form - I realization

Direct form - II

Given $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

By taking z-transforms on both sides

$$Y(z) = -0.1z^{-1}Y(z) + 0.2z^{-2}Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.2z^{-2}\right] = X(z)\left[3 + 3.6z^{-1} + 0.6z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

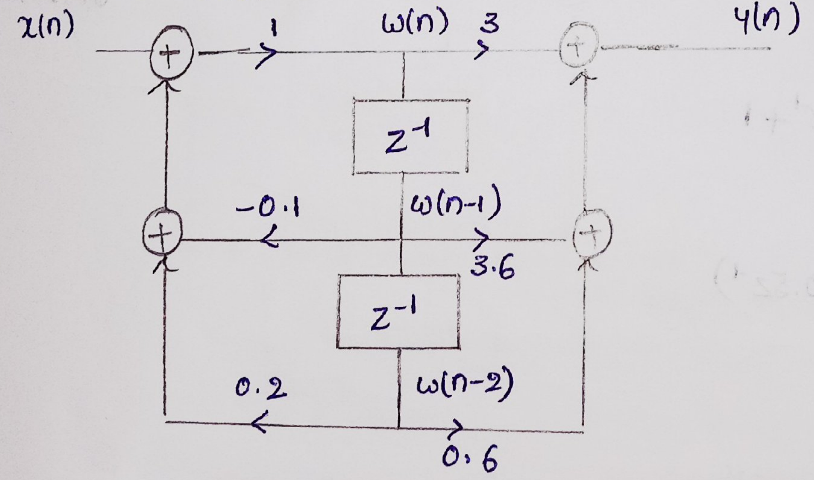


Fig: Direct form - II realization

Cascade form

We have $\dfrac{Y(z)}{X(z)} = \dfrac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.9z^{-2}}$

$$= \frac{3(1+1.2z^{-1}+0.2z^{-2})}{1+0.1z^{-1}-0.2z^{-2}}$$

$$= \frac{3(1+0.2z^{-1})(1+z^{-1})}{(1-0.4z^{-1})(1+0.5z^{-1})} = \frac{(3+0.6z^{-1})(1+z^{-1})}{(1-0.4z^{-1})(1+0.5z^{-1})}$$

$$= \frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

$H(z) = H_1(z).\ H_2(z)$

Let $H_1(z) = \dfrac{3+0.6z^{-1}}{1+0.5z^{-1}}$; $H_2(z) = \dfrac{1+z^{-1}}{1-0.4z^{-1}}$

Now, we realize $H_1(z)$ and $H_2(z)$ and cascade both to get realization of $H(z)$

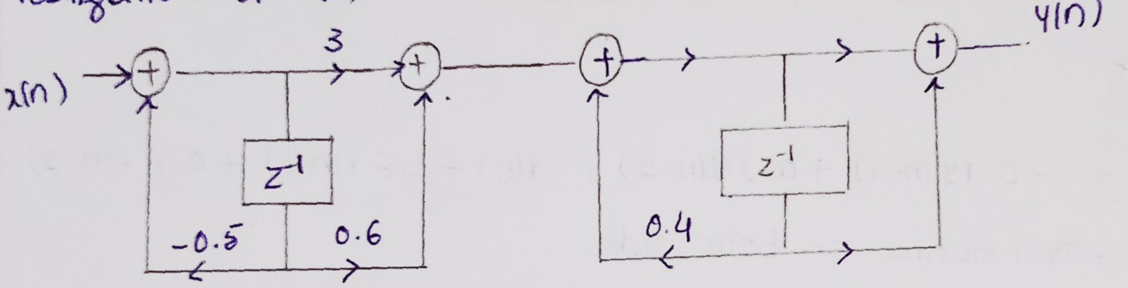


Fig: Cascade form realization

Parallel form

$$H(z) = \frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$

$$\begin{array}{r} -3 \\ -0.2z^{-2}+0.1z^{-1}+1 \enclose{longdiv}{0.6z^{-2}+3.6z^{-1}+3} \\ 0.6z^{-2}-0.3z^{-1}-3 \\ \hline 3.9z^{-1}+6 \end{array}$$

$$= -3 + \frac{3.9z^{-1}+6}{-0.2z^{-2}+0.1z^{-1}+1}$$

$$= -3 + \frac{6+3.9z^{-1}}{(1-0.4z^{-1})(1+0.5z^{-1})}$$

Let $H_1(z) = \dfrac{6+3.9z^{-1}}{(1-0.4z^{-1})(1+0.5z^{-1})} = \dfrac{A}{1-0.4z^{-1}} + \dfrac{B}{1+0.5z^{-1}}$.

$$A = \cancel{(1-0.4z^{-1})} \cdot \left.\frac{6+3.9z^{-1}}{\cancel{(1-0.4z^{-1})}(1+0.5z^{-1})}\right|_{z^{-1}=\frac{1}{0.4}}$$

$$= \frac{6+3.9\times\frac{1}{0.4}}{1+0.5\times\frac{1}{0.4}} = \frac{6+9.75}{1+1.25} = \frac{15.75}{2.25} = 7.$$

$$B = \cancel{(1+0.5z^{-1})} \cdot \left.\frac{6+3.9z^{-1}}{(1-0.4z^{-1})\cancel{(1+0.5z^{-1})}}\right|_{z^{-1}=\frac{-1}{0.5}}$$

$$= \cancel{(1+0.5z^{-1})}\ \frac{6+3.9\times\left(\frac{-1}{0.5}\right)}{1+0.4\times\frac{1}{0.5}} = \frac{6-7.8}{1+0.8} = \frac{-1.8}{1.8} = -1$$

$$\frac{6+3.9z^{-1}}{(1-0.4z^{-1})(1+0.5z^{-1})} = \frac{7}{1-0.4z^{-1}} - \frac{1}{1+0.5z^{-1}}.$$

$$\therefore\ H(z) = -3 + \frac{7}{1-0.4z^{-1}} - \frac{1}{1+0.5z^{-1}}$$

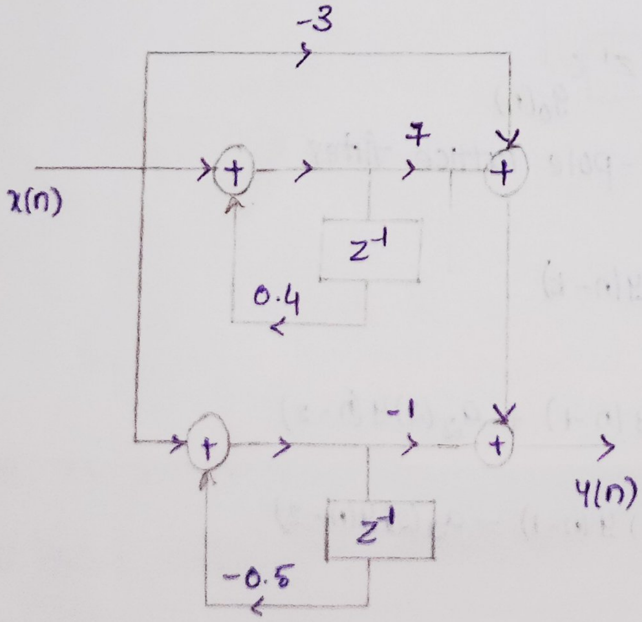


Fig: Parallel form realization of given example

→ Lattice structure of IIR system

Let us consider an all-pole system with system function

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N z^{-k}} = \frac{1}{A_N(z)}$$

The difference equation for this IIR system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_N z^{-k}}$$

$$Y(z)\left[1+\sum_{k=1}^{N} a_N z^{-k}\right] = X(z)$$

$$Y(z) + \sum_{k=1}^{N} a_N z^{-k} Y(z) = X(z)$$

By taking inverse z-transform to above equation

$$y(n) + \sum_{k=1}^{N} a_N y(n-k) = x(n)$$

For $N=1$

$$x(n) = y(n) + a_1(1)\, y(n-1)$$

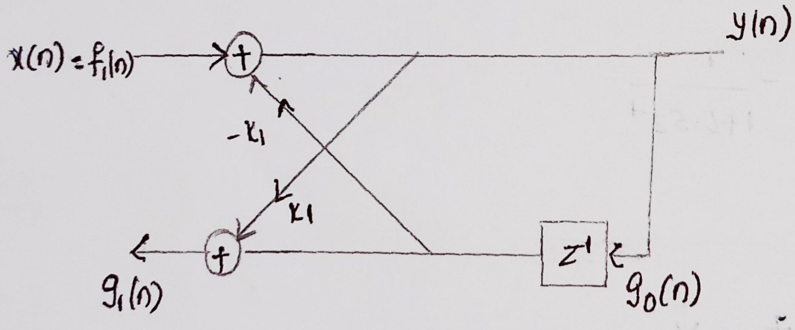


Fig: single stage all-pole lattice filter

For $N=2$ → $x(n) = y(n) + \sum_{k=1}^{2} a_N y(n-k)$

$$x(n) = y(n) + a_2(1)\, y(n-1) + a_2(2)\, y(n-2)$$

$$\therefore y(n) = x(n) - a_2(1)\, y(n-1) - a_2(2)\, y(n-2)$$

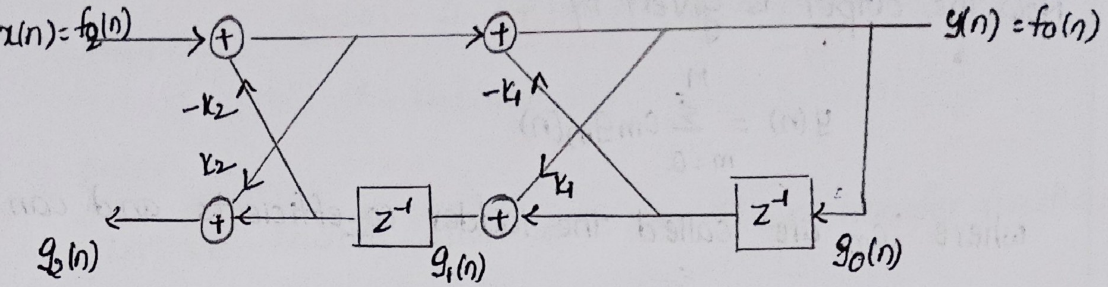


Fig : Two stage all-pole lattice filter

Conversion from Lattice structure to direct form

In general $a_m(0) = 1$; $a_m(k) = a_{m-1}(k) + a_m(m)\, a_{m-1}(m-k)$

$$a_m(m) = k_m$$

Conversion from direct form to Lattice structure

In general $a_{m-1}(0) = 1$; $k_m = a_m(m)$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$$

→ **Lattice - Ladder structure**

A general IIR filter containing both poles and zeros can be realized using an all pole lattice as the basic building block.

Consider an IIR filter with system function

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{M} b_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} \quad \text{—} ①$$

where $N \geq M$. A lattice structure for eq① can be constructed by first realizing an all-pole lattice co-efficients k_m, $1 \leq m \leq N$ for the denominator $A_N(z)$, and then adding a ladder part as shown in figure for $M = N$. The output of the ladder part can be expressed as a weighted linear combination of $\{g_m(n)\}$.

Now the output is given by

$$y(n) = \sum_{m=0}^{M} c_m g_m(n)$$

where 'c_m' are called the ladder co-efficients and can be obtained using the recursive relation

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m) \; ; \quad m = M, M-1, \ldots 0$$

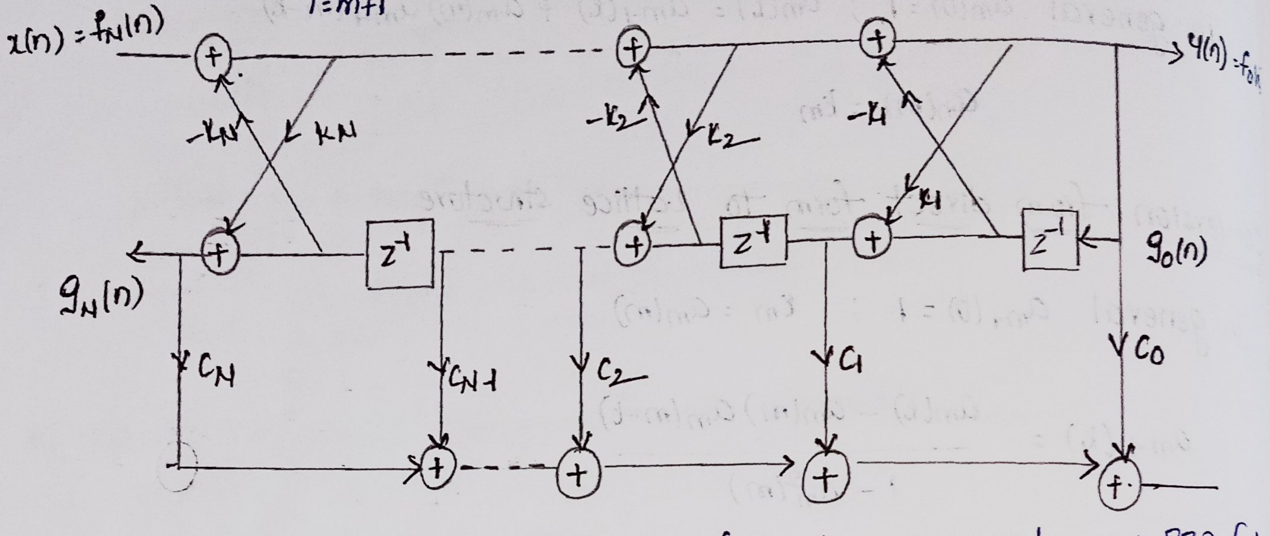


Fig: Lattice - Ladder structure for realizing a pole-zero IIR filter

→ Convert the following pole-zero IIR filter into a Lattice - ladder structure.

$$H(z) = \frac{1+2z^{-1}+2z^{-2}+z^{-3}}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}}$$

Sol Given $H(z) = \dfrac{1+2z^{-1}+2z^{-2}+z^{-3}}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}} = \dfrac{b_M(z)}{A_N(z)}$ — ①

$b_M(z) = 1+2z^{-1}+2z^{-2}+z^{-3}$ — ②

$A_N(z) = 1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}$ — ③

$a_3(0) = 1, \quad a_3(1) = \frac{13}{24}, \quad a_3(2) = \frac{5}{8}, \quad a_3(3) = \frac{1}{3}$

$k_3 = a_3(3) = \frac{1}{3}$

We know that $a_{m-1}(k) = \dfrac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$ — (4)

where 'm' is order

To construct the Lattice - Ladder structure to the given transfer function of order is 3. For that we require k_1, k_2, k_3

i.e $k_3 = a_3(3)$, $k_2 = a_2(2)$, $k_1 = a_1(1)$ is derived by above formula

For finding $a_2(2)$, We take $m=3$ and $k=2$

$$k_2 = a_2(2) = \frac{a_3(2) - a_3(3)\, a_3(1)}{1 - a_3^2(3)} = \frac{\frac{5}{8} - \frac{1}{3}\cdot\frac{13}{24}}{1-\left(\frac{1}{3}\right)^2}$$

$$= \frac{\frac{5}{8} - \frac{13}{72}}{1 - \frac{1}{9}} = \frac{\frac{45-13}{72}}{\frac{9-1}{9}} = \frac{\cancel{32}^{4}}{\cancel{72}_{8}} \times \frac{9}{\cancel{8}2} = \frac{1}{2} \quad — (5)$$

For finding $a_1(1)$, We take $m=2$, $k=1$

$$k_1 = a_1(1) = \frac{a_2(1) - a_2(2)\, a_2(1)}{1 - a_2^2(2)} \quad — (6)$$

Let us calculate $a_2(1)$ for that take $m=3, k=1$

$$a_2(1) = \frac{a_3(1) - a_3(3)\, a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{\frac{13}{24} - \frac{1}{3}\cdot\frac{5}{8}}{1 - \frac{1}{9}} = \frac{\frac{13}{24} - \frac{5}{24}}{\frac{8}{9}} = \frac{\cancel{8}}{\cancel{24}_{8}} \times \frac{\cancel{9}^{3}}{\cancel{8}} = \frac{3}{8} \quad — (7)$$

Substitute $a_2(1)$ in eq (6) We get

$$k_1 = a_1(1) = \frac{\frac{3}{8} - \frac{1}{2}\cdot\frac{3}{8}}{1 - \left(\frac{1}{2}\right)^2} = \frac{\frac{3}{8} - \frac{3}{16}}{1 - \frac{1}{4}} = \frac{\frac{6-3}{16}}{\frac{3}{4}} = \frac{\cancel{3}}{\cancel{16}_{4}} \times \frac{\cancel{4}}{\cancel{3}} = \frac{1}{4}$$

Therefore, for lattice structure

$$K_1 = \frac{1}{4}, \quad K_2 = \frac{1}{2}, \quad K_3 = \frac{1}{3}$$

For ladder structure

$$C_m = b_m - \sum_{i=m+1}^{M} C_i a_i (i-m) \quad ; \quad m = M, M-1, \ldots 0$$

$$C_3 = b_3 = 1.$$

The Numeration of given transfer function is

$$b_M(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

By comparing $b_3 = 1, \; b_2 = 2, \; b_1 = 2, \; b_0 = 1$

$$C_2 = b_2 - \sum_{i=3}^{3} C_i a_i (i-2)$$

$$= b_2 - [C_3 a_3(1)] = b_2 - C_3 a_3(1)$$

$$= 2 - 1\left(\frac{13}{24}\right) = 1.4583$$

$$C_1 = b_1 - \sum_{i=2}^{3} C_i a_i (i-1)$$

$$= b_1 - [C_2 a_2(1) + C_3 a_3(2)]$$

$$= 2 - \left[(1.4583)\left(\frac{3}{8}\right) + 1\left(\frac{5}{8}\right)\right]$$

$$= 0.8281$$

$$C_0 = b_0 - \sum_{i=1}^{3} C_i a_i (i-0)$$

$$= b_0 - [C_1 a_1(1) + C_2 a_2(2) + C_3 a_3(3)]$$

$$= 1 - \left[0.8281\left(\frac{1}{4}\right) + 1.4583\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right)\right]$$

$$= -0.2695$$

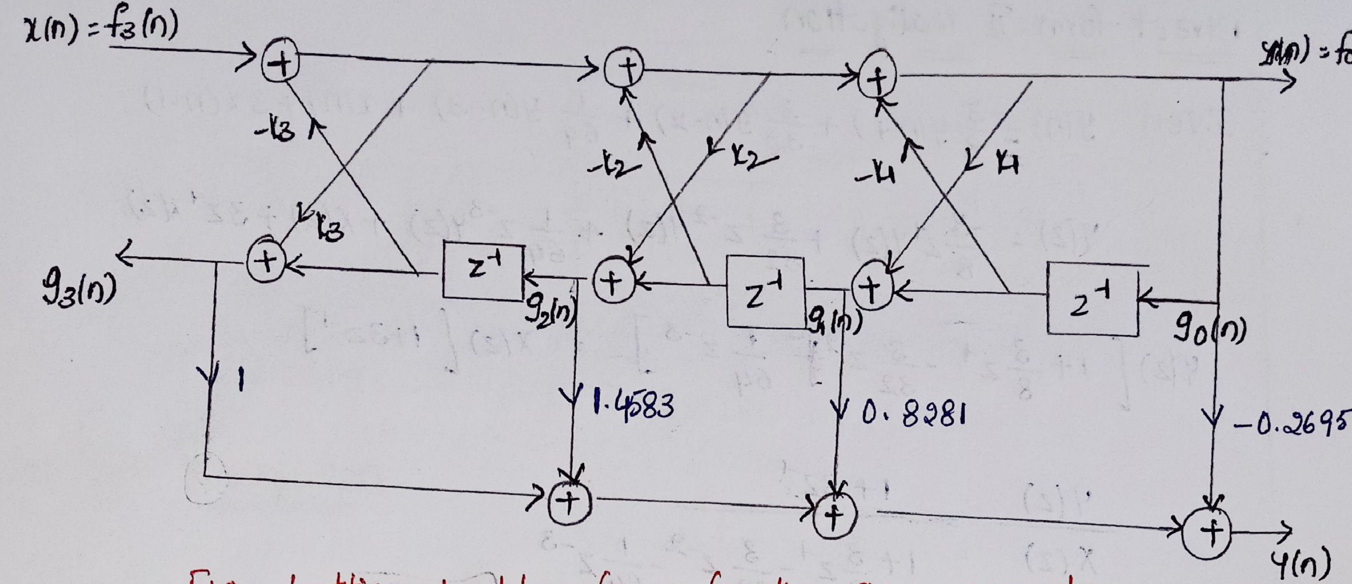


Fig: Lattice-Ladder form for the given example.

P.P. → obtain direct form I and direct form II realization of the LTI system governed by the equation.

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1)$$

Sol **Direct form-I Realization**

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1)$$

Let $x(n) + 3x(n-1) = \omega(n)$

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + \omega(n)$$

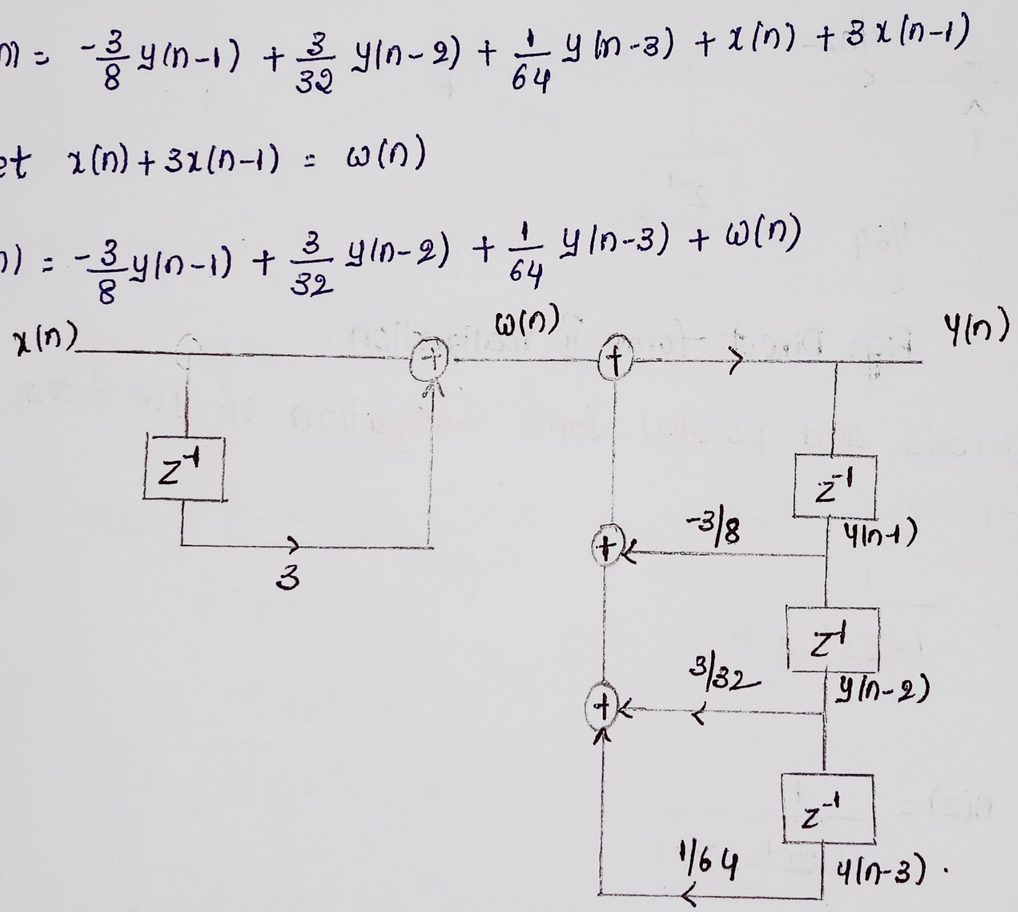


Fig: Direct form-I realization

Direct form-II realization

Given $y(n) = \frac{-3}{8} y(n-1) + \frac{3}{32} y(n-2) + \frac{1}{64} y(n-3) + x(n) + 3x(n-1)$

$$Y(z) = \frac{-3}{8} z^{-1} Y(z) + \frac{3}{32} z^{-2} Y(z) + \frac{1}{64} z^{-3} Y(z) + X(z) + 3z^{-1} X(z)$$

$$Y(z)\left[1 + \frac{3}{8} z^{-1} - \frac{3}{32} z^{-2} - \frac{1}{64} z^{-3}\right] = X(z)\left[1+3z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+3z^{-1}}{1+\frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}}$$

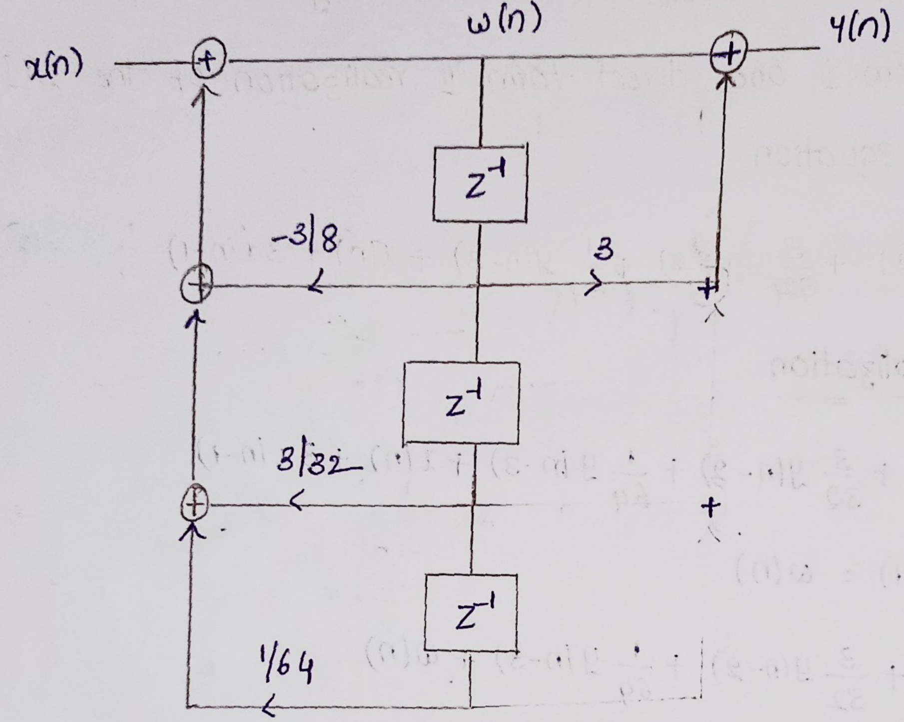


Fig: Direct form-II realization.

→ obtain the cascade and parallel form realization of IIR filter with system function

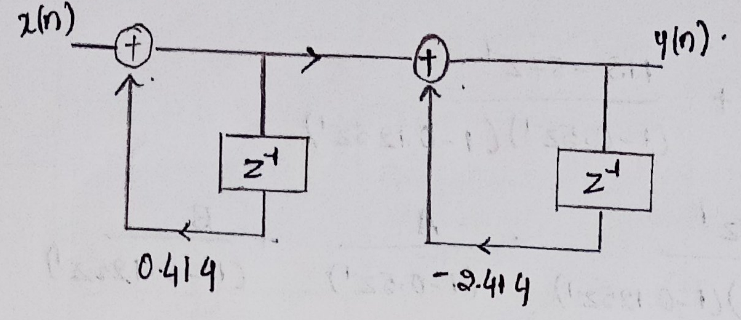
$$H(z) = \frac{1}{1+2z^{-1}-z^{-2}}.$$

sol. Cascade form

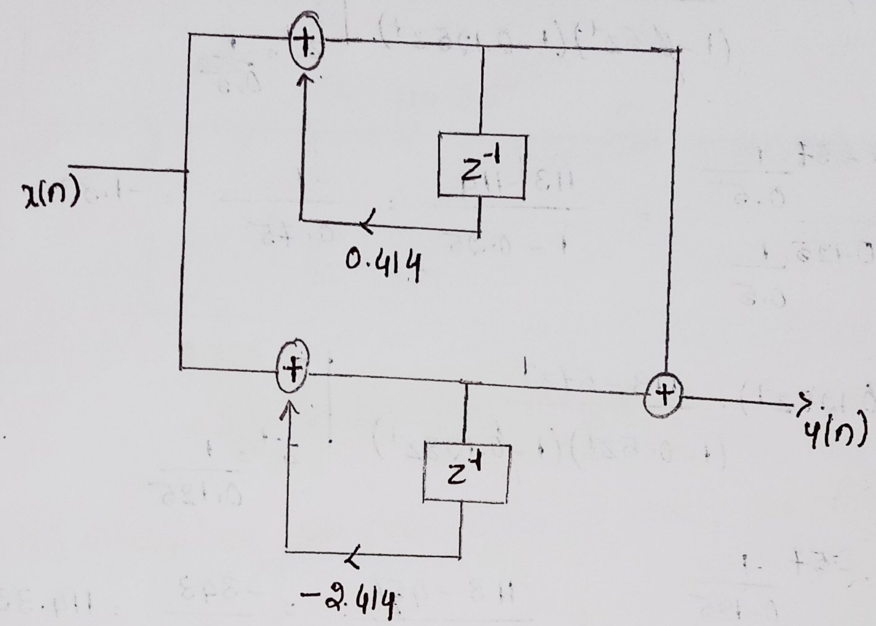
Given $H(z) = \frac{1}{1+2z^{-1}-z^{-2}}$

$$= \frac{1}{(1-0.414z^{-1})(1+2.414z^{-1})} = H_1(z).H_2(z)$$

$$H_1(z) = \frac{1}{(1-0.414z^{-1})} \qquad H_2(z) = \frac{1}{(1+2.414z^{-1})}$$



Parallel form Realization



→ Draw the parallel form realization of a system with system function

$$H(z) = \frac{(1-z^{-1})^3}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{8}z^{-1})}.$$

Sol Given $H(z) = \dfrac{(1-z^{-1})^3}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{8}z^{-1})}$

$$= \frac{1-z^{-3}-3z^{-1}+3z^{-2}}{1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-1}+\frac{1}{16}z^{-2}}$$

$$= \frac{1-3z^{-1}+3z^{-2}-z^{-3}}{1-\frac{5}{8}z^{-1}+\frac{1}{16}z^{-2}}$$

$$= \frac{1-3z^{-1}+3z^{-2}-z^{-3}}{1-0.625z^{-1}+0.0625z^{-2}}$$

$$\begin{array}{r} -16z^{-1}-112 \\ 0.0625z^{-2}-0.625z^{-1}+1 \overline{\smash{)}\,-z^{-3}+3z^{-2}-3z^{-1}+1} \\ \underline{-z^{-3}+10z^{-2}-16z^{-1}} \\ -7z^{-2}+13z^{-1}+1 \\ \underline{-7z^{-2}+70z^{-1}-112} \\ -57z^{-1}+113 \end{array}$$

$$= -16z^{-1}-112+\frac{-57z^{-1}+113}{0.0625z^{-2}-0.625z^{-1}+1}$$

$$H(z) = -16z^{-1} - 112 + \frac{113 - 57z^{-1}}{0.0625z^{-2} - 0.625z^{-1} + 1}$$

$$= -16z^{-1} - 112 + \frac{113 - 57z^{-1}}{(1 - 0.5z^{-1})(1 - 0.125z^{-1})}$$

Let $H_1(z) = \dfrac{113 - 57z^{-1}}{(1-0.5z^{-1})(1-0.125z^{-1})} = \dfrac{A}{(1-0.5z^{-1})} + \dfrac{B}{(1-0.125z^{-1})}$

$$A = (1 - \cancel{0}.5z^{-1}) \cdot \frac{113 - 57z^{-1}}{(1-\cancel{0}.5z^{-1})(1-0.125z^{-1})} \Bigg|_{z^{-1} = \frac{1}{0.5}}$$

$$= \frac{113 - 57 \cdot \frac{1}{0.5}}{1 - 0.125 \cdot \frac{1}{0.5}} = \frac{113 - 114}{1 - 0.25} = \frac{-1}{0.75} = -1.33$$

$$B = (1 - \cancel{0}.125z^{-1}) \cdot \frac{113 - 57z^{-1}}{(1-0.5z^{-1})(1-\cancel{0}.125z^{-1})} \Bigg|_{z^{-1} = \frac{1}{0.125}}$$

$$= \frac{113 - 57 \cdot \frac{1}{0.125}}{1 - 0.5 \times \frac{1}{0.125}} = \frac{113 - 456}{1 - 4} = \frac{-343}{-3} = 114.33$$

$$\therefore \frac{113 - 57z^{-1}}{(1-0.5z^{-1})(1-0.125z^{-1})} = \frac{-1.33}{(1-0.5z^{-1})} + \frac{114.33}{(1-0.125z^{-1})}$$

$$\therefore H(z) = -112 - 16z^{-1} - \frac{1.33}{1-0.5z^{-1}} + \frac{114.33}{(1-0.125z^{-1})}$$

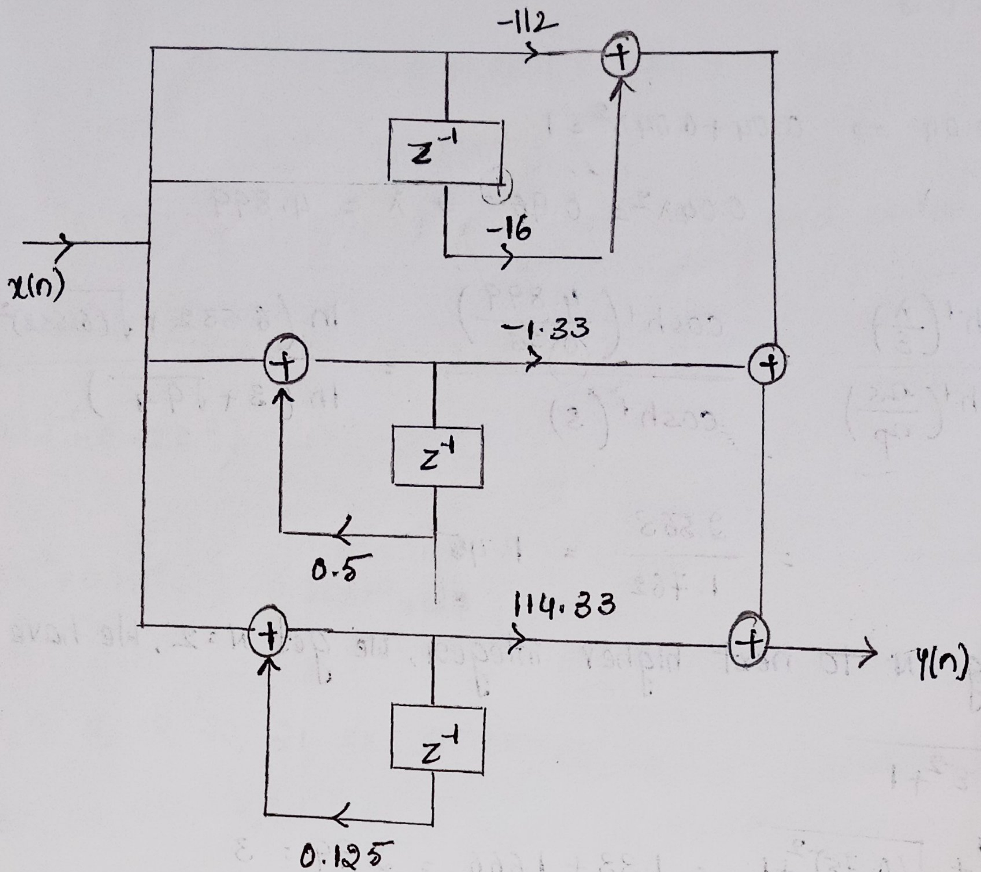


Fig: Parallel form realization

→ Design a digital chebyshev low pass filter to satisfy the following constraints

$0.8 \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi$

$0 \le |H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi.$

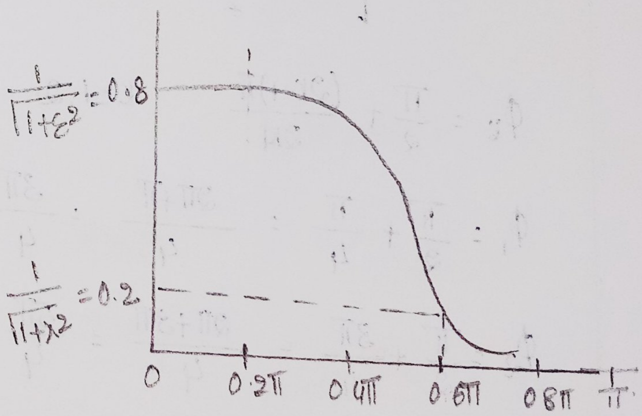
using impulse invarient transformation and assume $T = 1sec$.

Sol. By using impulse invariance method

$\omega = \Omega T \Rightarrow \omega_p = \Omega_p T$ and $\omega_s = \Omega_s T$

Given $T = 1sec$

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$



$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}.$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \frac{1}{1+\varepsilon^2} = 0.64$$

$$0.64 + \varepsilon^2 0.64 = 1 \Rightarrow \varepsilon^2 0.64 = 0.36 \Rightarrow \varepsilon^2 = \frac{0.36}{0.64} \Rightarrow \varepsilon = 0.75$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\frac{1}{1+\lambda^2} = 0.04 \Rightarrow 0.04 + 0.04\lambda^2 = 1$$

$$0.04\lambda^2 = 0.96, \Rightarrow \lambda = 4.899$$

$$N = \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega s}{\Omega p}\right)} = \frac{\cosh^{-1}\left(\frac{4.899}{0.75}\right)}{\cosh^{-1}(3)} = \frac{\ln\left(6.532 + \sqrt{(6.532)^2 - 1}\right)}{\ln\left(3 + \sqrt{9-1}\right)}$$

$$= \frac{2.563}{1.762} = 1.45$$

Approximating N to next higher integer, we get $N = 2$, We have

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$$

$$= (0.75)^{-1} + \sqrt{(0.75)^{-2} + 1} = 1.33 + 1.666 = 2.99 = 3$$

$$a = \Omega p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$= 0.2\pi \left[\frac{3^{1/2} - 3^{-1/2}}{2}\right] = 0.2\pi \left[\frac{1.732 - 0.577}{2}\right] = 0.3628$$

$$b = \Omega p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 0.2\pi \left[\frac{1.732 + 0.577}{2}\right] = 0.725$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{2\pi + \pi}{4} = \frac{3\pi}{4} = 135^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{2\pi + 3\pi}{4} = \frac{5\pi}{4} = 225^\circ$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1$$

$$= 0.3628\cos\left(\frac{3\pi}{4}\right) + j0.725\sin\left(\frac{3\pi}{4}\right)$$

$$= 0.3628(-0.707) + j0.725(0.707)$$

$$= -0.256 + j0.513$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2$$

$$= 0.3628\cos\left(\frac{5\pi}{4}\right) + j0.725\sin\left(\frac{5\pi}{4}\right)$$

$$= 0.3628(-0.707) + j0.725(-0.707)$$

$$= -0.256 - j0.513$$

Denominator of $H(s) = (s+0.256)^2 + (0.513)^2$

$$= s^2 + 0.0655 + 0.512s + 0.263$$

$$= s^2 + 0.512s + 0.33$$

For N is even, put $s=0$ in denominator and divide it by $\sqrt{1+\varepsilon^2}$

$$\sqrt{1+(0.75)^2} = 1.25$$

Numerator of $H(s) = \frac{0.33}{1.25} = 0.264$

$$\therefore \quad H(s) = \frac{0.264}{s^2+0.512s+0.33} = \frac{0.264}{(s+0.256-j0.513)(s+0.256+j0.513)}.$$

$$\frac{0.264}{(s+0.256-j0.513)(s+0.256+j0.513)} = \frac{A}{(s+0.256-j0.513)} + \frac{B}{(s+0.256+j0.513)}.$$

$$A = \frac{0.264\,\cancel{(s+0.256-j0.513)}}{\cancel{(s+0.256-j0.513)}(s+0.256+j0.513)}\Bigg|_{s=-0.256+j0.513}$$

$$= \frac{0.264}{-0.256+j0.513+0.256+j0.513} = \frac{0.264}{j(1.026)} \times \frac{j}{j} = -j0.257.$$

$$B = \frac{(0.264)\cancel{(s+0.256+j0.513)}}{(s+0.256-j0.513)\cancel{(s+0.256+j0.513)}}\Bigg|_{s=-0.256-j0.513}$$

$$= \frac{0.264}{-0.\cancel{256}-j0.513+0.\cancel{256}-j0.513} = \frac{0.264}{-j1.026} \times \frac{j}{j} = +j0.257.$$

$$\therefore \frac{0.264}{s^2+0.\cancel{256}512s+0.33} = \frac{-j0.257}{s+0.256-j0.513} + \frac{j0.257}{s+0.256+j0.513}$$

$$= \frac{-j0.257}{s-(-0.256+j0.513)} + \frac{j0.257}{s-(-0.256-j0.513)}$$

By using Linear invarient Method

If $H(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$ then $H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

The poles are $P_1 = -0.256 + j0.513$

$P_2 = -0.256 - j0.513$

$$\therefore H(z) = \frac{-0.257j}{1 - e^{-0.256T} e^{j0.513T} z^{-1}} + \frac{0.257j}{1 - e^{-0.256T} e^{-j0.513T} z^{-1}}$$

Assume T = 1sec

$$H(z) = \frac{0.257j}{1 - e^{-0.256} e^{-j0.513} z^{-1}} - \frac{0.257j}{1 - e^{-0.256T} e^{j0.513} z^{-1}}.$$

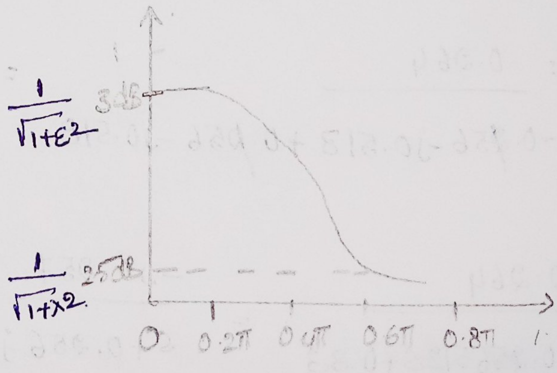
$$H(z) = \frac{0.1954 z^{-1}}{1 - 1.3483 z^{-1} + 0.5987 z^{-2}}.$$

→ Determine the system function H(z) of the lowest order chebyshev digital filter with the following specification

(a) 3dB ripple in passband $0 \le \omega \le 0.2\pi$

(b) 25dB attenuation in stopband $0.45\pi \le \omega \le \pi$.

Sol Given



$\omega_p = 0.2\pi$; $\omega_s = 0.45\pi$

$\alpha_p = 3dB$; $\alpha_s = 25dB$; T = 1

$\frac{1}{\sqrt{1+\epsilon^2}} = 3$ $\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 0.65$

$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 1.71$ $\epsilon = \sqrt{10^{0.1\alpha_p} - 1}$

$\frac{1}{1+\epsilon^2} = 9 \Rightarrow 9 + 9\epsilon^2 = 1$

$9\epsilon^2 =$ $= \sqrt{10^{0.3} - 1} = 1$

$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1+\sqrt{1+1} = 1+\sqrt{2} = 1+1.414 = 2.414$

$a = \Omega p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$

$= 0.65 \left[\frac{(2.414)^{1/3} - (2.414)^{-1/3}}{2} \right]$

$= 0.65 \left[\frac{1.341 - 0.745}{2} \right]$

$= 0.65 \, (0.298) = 0.1937$

$b = \Omega p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$

$= 0.65 \left[\frac{(2.414)^{1/3} + (2.414)^{-1/3}}{2} \right]$

$= 0.65 \left(\frac{1.341 + 0.745}{2} \right]$

$= 0.65 \times 1.043 = 0.678$

$N \geq \frac{\cosh^{-1} \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)^{1/2}}{\cosh^{-1} \left(\frac{\Omega s}{\Omega p} \right)}$

$N \geq \frac{\cosh^{-1} \left(\frac{10^{0.1 \times 25} - 1}{10^{0.1 \times 3} - 1} \right)^{1/2}}{\cosh^{-1} (\Omega s / \Omega p)}$

$N \geq \frac{\cosh^{-1} \left(\frac{10^{2.5} - 1}{10^{0.3} - 1} \right)^{1/2}}{\cosh^{-1} \left(\frac{1.71}{0.65} \right)}$

$N \geq \frac{\cosh^{-1} \left(\frac{315.227}{0.99} \right)^{1/2}}{\cosh^{-1} (2.63}$

$N \geq \frac{\cosh^{-1} (17.8)}{\cosh^{-1} (2.63)} = \frac{3.57}{1.62} = 2.2.$

$\therefore N = 3$

$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2,3$

$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi + \pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3} = 120°.$

$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} = \frac{3\pi + 3\pi}{6} = \frac{6\pi}{6} = \pi = 180°.$

$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = \frac{3\pi + 5\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3} = 240°.$

$s_1 = a\cos\phi_1 + jb\sin\phi_1$

$= 0.1937 \cos(120°) + j\,0.678 \sin(120°)$

$= 0.1937 \times -0.5 + j\,0.678\,(0.866)$

$= -0.09685 + j\,0.587$

$s_2 = a\cos\phi_2 + jb\sin\phi_2$

$= 0.1937 \cos(180°) + j(0.678)\sin(180)$

$= -0.1937$

$S_3 = a\cos\phi_3 + jb\sin\phi_3$.

$= 0.1935\cos\left(\frac{4\pi}{3}\right) + j\,0.678\sin\left(\frac{4\pi}{3}\right)$

$= 0.1935\,(-0.5) + j\,0.678\,(-0.866)$

$= -0.09675 - j0.587$.

The denominator polynomial of $H(s) = (s+0.1937)\left[(s+0.09675)^2 + (0.587)^2\right]$

$= (s+0.1935)\left[s^2 + 0.1935s + 0.354\right]$

For N is odd, the numerator polynomial of $H(s)$ is put $s=0$

$0.1935(0.354) = 0.0685$.

The transfer function $H(s) = \dfrac{0.0685}{(s+0.1935)(s^2+0.1935s+0.354)}$.

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.0685}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1935\right)\left(\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.1935\times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.354\right]}$$

$$= \frac{0.0685(1+z^{-1})^3}{(2-2z^{-1}+0.1935+0.1935z^{-1})\left[4(1+z^{-2}-2z^{-1}) + 0.387(1-z^{-1})(1+z^{-1}) + 0.354(1+z^{-2}\ldots\right.}$$

$$= \frac{0.0685(1+z^{-1})^3}{(2.1935-1.8065z^{-1})\left[4+4z^{-2}-8z^{-1}+0.387(1-z^{-2})+0.354(1+z^{-2}+2z^{-1})\right)}$$

$$= \frac{0.0685(1+z^{-1})^3}{(2.1935-1.8065z^{-1})\left[3.967z^{-2}-7.292z^{-1}+4.741\right]}$$

$$= \frac{0.00658(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.538z^{-1}+0.835z^{-2})}$$

$$\frac{0.0687(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.16z^{-1}+0.915z^{-2})}$$

→ Design a digital Butterworth band pass filter using Bilinear transformation for the following specifications:

Lower stop band edge : 25Hz Lower pass band edge : 100Hz

upper stop band edge : 225Hz Upper pass band edge : 150Hz

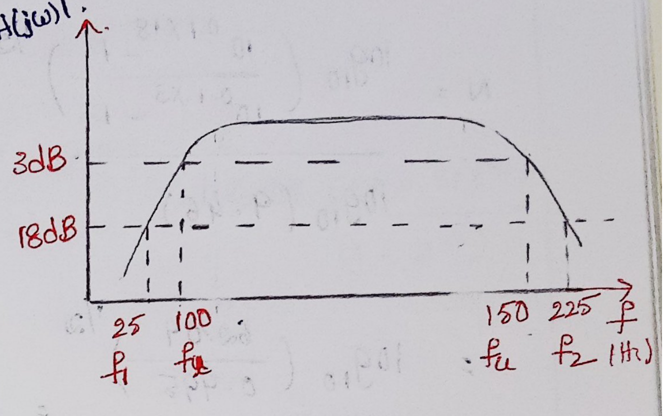
stop band attenuation = 18dB ; Pass band ripple : 3dB

Sampling frequency = 500Hz.

Sol. Given $f_1 = 25Hz$ $f_2 = 225Hz$

$f_l = 100Hz$ $f_u = 150Hz$

sampling frequency $f = 500Hz$.



$$\frac{\omega_1 T}{2} = \frac{2\pi f_1}{2f} = \frac{\cancel{2}\pi \times \cancel{25}}{\cancel{2}\times \cancel{500}_{20}} = \frac{\pi}{20}$$

$$\frac{\omega_l T}{2} = \frac{2\pi f_l}{2f} = \frac{2\pi \times 100}{2\times 500} = \frac{\pi}{5}$$

$$\frac{\omega_u T}{2} = \frac{2\pi f_u}{2f} = \frac{\cancel{2}\pi \times \cancel{150}^{3}}{\cancel{2}\times \cancel{500}_{10}} = \frac{3\pi}{10}$$

$$\frac{\omega_2 T}{2} = \frac{2\pi f_2}{2f} = \frac{\cancel{2}\pi \times \cancel{225}^{\cancel{45}\,9}}{\cancel{2}\times \cancel{500}_{\cancel{100}\,20}} = \frac{9\pi}{20}$$

Prewrapped analog frequencies

$$\frac{\Omega_1 T}{2} = \tan\left(\frac{\omega_1 T}{2}\right) = \tan\left(\frac{\pi}{20}\right) = 0.1584$$

$$\frac{\Omega_l T}{2} = \tan\left(\frac{\omega_l T}{2}\right) = \tan\left(\frac{\pi}{5}\right) = 0.727$$

$$\frac{\Omega_u T}{2} = \tan\left(\frac{\omega_u T}{2}\right) = \tan\left(\frac{3\pi}{10}\right) = 1.376$$

$$\frac{\Omega_2 T}{2} = \tan\left(\frac{\omega_2 T}{2}\right) = \tan\left(\frac{9\pi}{20}\right) = 6.313$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} = \frac{-(0.1584)^2 + 0.727\times 1.376}{0.1584(1.376 - 0.727)} = \frac{-0.026+1}{0.1584(0.65)} = \frac{0.975}{0.103}$$

$$= 9.46$$

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = \frac{(6.313)^2 - 0.727 \times 1.376}{6.313(1.376 - 0.727)}$$

$$= \frac{39.853 - 1}{6.313(0.65)} = \frac{38.853}{4.10} = 9.476$$

$$\Omega_r = \min\{|A|, |B|\} = 9.46$$

$$N = \frac{\log_{10}\left(\frac{10^{0.1 \times 18} - 1}{10^{0.1 \times 3} - 1}\right)^{1/2}}{\log_{10}(9.46)}$$

$$= \frac{\log_{10}\left(\frac{62.09}{0.995}\right)^{1/2}}{\log_{10}(9.46)} = \frac{0.89}{0.975} = 0.917 \Rightarrow N = 1$$

The transfer function for order $N=1$ is

$$H(s) = (s+1)$$

The transformation for the BPF is

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 0.727 \times 1.376}{s(1.376 - 0.727)} = \frac{s^2 + 1}{s(0.649)}$$

$$H(s) = \frac{s^2 + 1}{0.649s} + 1. = \frac{s^2 + 0.649s + 1}{0.649s}$$

By using Bilinear transformation

$$H(z) = H(s)\Big|_{\frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.649\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}{0.649\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{\frac{(1-z^{-1})^2 + 0.649(1-z^{-1})(1+z^{-1}) + (1+z^{-1})^2}{(1+z^{-1})^2}}{\frac{0.649(1-z^{-1})}{(1+z^{-1})}}$$

$$= \frac{1+z^{-2}-\cancel{2z^{-1}}+0.649-0.649z^{-2}+1+z^{-2}+\cancel{2z^{-1}}}{0.649(1-z^{-2})}$$

$$= \frac{1.351z^{-2}+2.649}{0.649(1-z^{-2})} = \frac{2.649(1+0.51z^{-2})}{0.649(1-z^{-2})} = 4.08\frac{(1+0.51z^{-2})}{(1-z^{-2})}$$

$$\therefore H(z) = \frac{4.08(1+0.51z^{-2})}{(1-z^{-2})}$$

→ **Frequency Transformation in Digital Domain**

A digital low pass filter can be converted into a digital high pass, band stop, band pass or another digital filter. These transformation are given below:

Lowpass to Lowpass:

$$z^{-1} = \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \qquad \text{where } \alpha = \frac{\sin[(\omega p - \omega p')/2]}{\sin[(\omega p + \omega p')/2]}$$

ωp = passband frequency of LPF

$\omega p'$ = passband frequency of new LPF

Lowpass to Highpass:

$$z^{-1} = -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right] \qquad \text{where } \alpha = -\frac{\cos[(\omega p' + \omega p)/2]}{\cos[(\omega p' - \omega p/2]}$$

ωp = passband frequeny of LPF

$\omega p'$ = passband frequeny of HPF

Lowpass to Bandpass

$$z^{-1} \rightarrow \frac{-\left(z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right)}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$

where $\alpha = \frac{\cos((\omega_p + \omega_l)/2)}{\cos[(\omega_u - \omega_l)/2]}$; $k = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan \frac{\omega_p}{2}$

ω_u = upper cut off frequency

ω_l = lower cut-off frequency

Lowpass to Bandstop

$$z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$$

where $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$; $k = \tan\left[\frac{(\omega_u - \omega_l)}{2}\right] \tan \frac{\omega_p}{2}$